On the Most Visited Sites of Symmetric Markov Processes*

Nathalie Eisenbaum
Université de Paris, VI

Davar Khoshnevisan†
University of Utah

July 24, 2002

Abstract

A growing body of recent works have been devoted to the study of the favorite points of various concrete Markov processes. We contribute to this subject by showing that for a large class of recurrent strongly symmetric Markov processes, singletons are polar for the most visited site(s).

Keywords most visited site, local times, symmetric Markov processes.

AMS 2000 subject classification 60J25, 60J55.

1 Introduction

Given a simple symmetric random walk S with $S_0 = 0$, *most visited sites* (or *favorite points*) $\Xi = \{\Xi_1, \Xi_2, \ldots\}$ are defined as

$$\Xi_n = \Big\{ x \in \mathbb{Z} : \sum_{j=1}^{n} \mathbf{1}_{\{S_j = x\}} = \sup_{y \in \mathbb{Z}} \sum_{i=1}^{n} \mathbf{1}_{\{S_i = y\}} \Big\}, \qquad \forall n = 0, 1, \ldots.$$

The process $\Xi = \{\Xi_n;\, n \geq 1\}$ was first introduced and studied in Erdős and Révész (1984) who, amongst other things, showed that any element of Ξ_n satisfies Khintchine's law of the iterated logarithm as $n \to \infty$. Subsequently, Bass and Griffin (1985) answered a question of Erdős and Révész (1984) by proving that

$$\lim_{n \to \infty} \inf_{x \in \Xi_n} |x| = +\infty, \qquad \text{a.s.}, \tag{1.1}$$

In particular, from (1.1) one deduces the surprising fact that the favorite points of S are transient.

The mentioned results have spurred a good deal of recent activity in the subject; cf. Shi and Tóth (2000) for a recent survey. In this regard, we also mention a recent preprint of Lifshits and Shi (2001) where the precise rate of explosion in (1.1) is presented; this solves a long-standing open problem in this area.

To solve the mentioned problem of Erdős and Révész, Bass and Griffin (1985) first approximate Ξ_n by the most visited sites of a linear Brownian motion B with $B_0 = 0$. Then, they derive a continuous-time version of (1.1) by showing that almost surely, $\lim_{t\to\infty} \inf_{x \in \Upsilon_t} |x| = +\infty$, where

*Research partially supported by a grant from the North Atlantic Treaty Organization
†Research partially supported by grants from the National Science Foundation

Υ denotes the most visited sites of B as defined by $\Upsilon_t = \{x \in \mathbb{R} : \ell_t^x = \sup_{y\in\mathbb{R}} \ell_t^y\}$, $\forall t \geq 0$. Here, $\ell_t^x = \int_0^t \delta_x(B_s)\,ds$ is Brownian local time. We refer the reader to Revuz and Yor (1991, Ch. VI) for a pedagogic treatment of local times.

In general, if X is any recurrent Hunt process on some state space E, and with continuous local times L, one defines its most visited sites (or *favorite points*) by

$$\mathcal{V}_t = \Big\{x \in E : \ L_t^x = \sup_{y\in E} L_t^y\Big\}. \tag{1.2}$$

Bass et al (2000) have shown that when X is a symmetric stable process of index $\alpha > 1$, and $E = \mathbb{R}$, its favorite points are transient, i.e., $\lim_{t\to\infty} \inf_{x\in\mathcal{V}_t} |x| = +\infty$, a.s. This has been extended to a larger class of Lévy processes by Marcus (2000). Little is known about the other properties of the favorite points of Markov processes.

We say that a given compact set $K \subset E$ is *polar* for $\mathcal{V}$, if

$$\mathbb{P}_{\mathbf{o}}\{\exists t > 0 : K \cap \mathcal{V}_t \neq \varnothing\} = 0. \tag{1.3}$$

Here, $\mathbf{o} \in E$ is some distinguished point that we hold fixed throughout, and $\mathbb{P}_{\mathbf{o}}$ is the law of X, given that $X_0 = \mathbf{o}$. We are interested in knowing when K is polar for the favorite points of a Markov process X. Eisenbaum (1997) has made progress toward this problem by showing that if X is a symmetric stable process of index $\alpha > 1$, then $\{\mathbf{o}\}$ is polar for $\mathcal{V}$. Equivalently, all singletons are polar for the most visited sites process. Here, we show that such a polarity result holds in greater generality. To describe our result, consider

$$T_{\mathbf{o}} = \inf\{t > 0 : \ X_t = \mathbf{o}\}. \tag{1.4}$$

In this way, we define the kernel

$$g(x, y) = \mathbb{E}_x\{L_{T_{\mathbf{o}}}^y\}, \qquad \forall x, y \in E. \tag{1.5}$$

This is the potential kernel for the process X killed upon reaching $\{\mathbf{o}\}$. Now, suppose that g is a symmetric function. It is then possible to show that the function g defined in (1.5) is also positive definite; cf. Eisenbaum (2002). As such, g is the covariance function of some centered Gaussian process $\eta = \{\eta_x;\ x \in E\}$ that we call the Gaussian process *associated* to X. For simplicity, we introduce η on the same probability space as X and assume that X and η are independent. This can always be done by considering product spaces in a standard way.

Since L is assumed to be continuous, so is η; cf. Marcus and Rosen (1992, Th. 1). We say that the associated process η has a *local envelope* at $\mathbf{o}$, if there exists a nonrandom function $\varphi : E \to \mathbb{R}_+$, and a countable sequence $\{x_n\}_{n\geq 1} \subset E$, such that

(LE-I) $\lim_{n\to\infty} x_n = \mathbf{o}$, whereas $0 = \varphi(\mathbf{o}) < \varphi(x_n)$ for all $n \geq 1$; and

(LE-II) with probability one, $\limsup_{n\to\infty} \dfrac{\eta_{x_n}}{\varphi(x_n)} = 1$.

Remark 1.1 Roughly speaking, η has a local envelope at $\mathbf{o}$ if it satisfies a kind of law of the iterated logarithm at $\mathbf{o}$, at least along a given subsequence. For example, consider $E = \mathbb{R}$, $\mathbf{o} = 0$. Then, if η were Brownian motion, it would have a local envelope at $\mathbf{o}$ along the sequence $x_n = 2^{-n}$ with $\varphi(x) = \sqrt{2x|\ln x|}$. It would also have a local envelope at $\mathbf{o}$ along the sequence $x_n = n^{-1}$ with $\varphi(x) = \sqrt{2x|\ln|\ln(x)||}$. □

Remark 1.2 The existence of a local envelope is not a trivial condition, as can be seen by considering $\eta_x = xZ$ ($x \in \mathbb{R} \equiv E$), where Z is a standard normal variate. [It can be shown that the latter process η is associated to a symmetric diffusion.] $\square$

The main result of this paper is

Theorem 1.3 *Let $\eta = \{\eta_x;\ x \in E\}$ be the Gaussian process associated to X and suppose that $\mathbb{P}_{\mathbf{o}}$-a.s., η has a local envelope at $\mathbf{o}$. Then, $\mathbf{o}$ is polar for $\mathcal{V}$.*

Showing that η has a local envelope is tantamount to verifying a local law of the iterated logarithm for η. As such, we can find a sufficient condition, as the following shows.

Proposition 1.4 *Suppose there exist $x_1, x_2, \ldots \in E$ such that $\lim_{n\to\infty} x_n = \mathbf{o}$ and*

$$\lim_{k\to\infty} (\ln k)^{\frac{1}{2}} \sup_{\substack{n,m\in\mathbb{N}:\\ |n-m|\geq k}} \frac{g(x_n, x_m)}{[g(x_n, x_n)\, g(x_m, x_m)]^{\frac{1}{2}}} = 0.$$

Then, $\mathbf{o}$ is polar for $\mathcal{V}$.

This will show that for a large class of Lévy processes, singletons are polar for the most visited sites process; see Theorem 5.2 below. Such Lévy processes include symmetric stable processes of index $\alpha > 1$ on $\mathbb{R}$.

To conclude the introduction, let us mention that we have not succeeded in resolving the following question that is motivated by a suggestion of an anonymous referee.

Open Problem 1.5 Are there recurrent Markov processes with local times, whose g-function is symmetric, and such that $\mathbf{o}$ is not polar for the most visited site? More specifically, is there a linear, symmetric, and recurrent Lévy process X, such that X possesses local times, and 0 is non-polar for the most visited sites? $\square$

Acknowledgement Portions of this work were carried out while the authors were visiting Ecole Polytechnique Fédérale de Lausanne, and while D. Kh. was visiting Université de Paris VI. We thank both institutions for their hospitality. We are also happy to thank an anonymous referee for his/her very careful reading of this paper, and for making a number of useful suggestions that have led to an improved end product.

2 The Associated Gaussian Process

The Gaussian process *associated* to X (in the sense of Theorem 1.3) is the process that naturally arises in the Dynkin's isomorphism theorem and its variants. We begin by introducing some notation, all the time remembering that $\mathbf{o} \in E$ is held fixed. Throughout, $\tau = \{\tau(t);\ t \geq 0\}$ stands for the *inverse local time process of X at* $\mathbf{o}$, defined as

$$\tau(t) = \inf\{t > 0 :\ L_t^{\mathbf{o}} > t\}, \qquad \forall t \geq 0. \tag{2.1}$$

According to Eisenbaum et al (2000, Th.1.1), for any cylinder function $F : E^{\mathbb{R}_+} \to \mathbb{R}_+$, and for all $t \geq 0$,

$$\mathbb{E}_{\mathbf{o}}\big\{F\big(L^{\bullet}_{\tau(t)} + \tfrac{1}{2}\eta^2_{\bullet}\big)\big\} = \mathbb{E}_{\mathbf{o}}\big\{F\big(\tfrac{1}{2}(\eta_{\bullet} + \sqrt{2t})^2\big)\big\}. \tag{2.2}$$

[To be precise, Eisenbaum et al (2000) verifies this under the extra condition of strong symmetry of X. However, this result continues to hold as long as the function g is symmetric; cf. Eisenbaum

(2002).] This is a case of an *isomorphism theorem* relating local times of Markov processes to Gaussian processes. An important consequence of this isomorphism theorem is that $x \mapsto \eta_x$ has a continuous modification, which we continue to write as η. This has been alluded to in §1 and is equivalent to, Eq. (2.2) and the continuity of $(t, x) \mapsto L^x_t$; cf. MARCUS AND ROSEN (1992, p. 1664) for an argument in a similar setting.

A key step in our proof of Theorem 1.3 is a weak convergence result that may be of independent interest. For all $\lambda > 0$, we define the stochastic process $Y_\lambda = \{Y_\lambda(x,t);\ x \in E,\ t \geq 0\}$ as

$$Y_\lambda(x,t) = \lambda^{-\frac{1}{2}}\big[L^x_{\tau(\lambda t)} - \lambda t\big]. \tag{2.3}$$

We are interested in deriving asymptotic results for the process Y_λ, as $\lambda \to +\infty$. In order to properly describe these asymptotics, we introduce the spaces on which various modes of weak convergence shall take place.

For any compact $K \subset E$ and for each fixed $T > 0$, let $\mathcal{D}_T(C(K))$ denote the Skorohod space of cadlag functions $f : [0,T] \to C(K)$ with $f(0) = 0$. Here, $C(K)$ denotes the space of all real continuous functions on K, and is endowed with the compact-open topology, i.e., the topology of uniform convergence. We also endow $\mathcal{D}_T(C(K))$ with the corresponding Skorohod topology; see ETHIER AND KURTZ (1986; Ch. 3) for details. Weak convergence on $\mathcal{D}_T(C(K))$ is denoted by $\xrightarrow{\mathcal{D}_T(C(K))}$, whereas weak convergence in $C(K)$ is denoted by $\xrightarrow{C(K)}$.

It is important to recognize that $K \times T \ni (x,t) \mapsto Y_\lambda(x,t)$ is a process in $\mathcal{D}_T(C(K))$ for any compact $K \subset E$. That is, $t \mapsto Y_\lambda(x,t)$ is cadlag, whereas $K \ni x \mapsto Y_\lambda(x,t)$ is in $C(K)$.

Theorem 2.1 *Fix some $T > 0$ and a compact $K \subset E$. Then, as $\lambda \to +\infty$, $Y_\lambda \xrightarrow{\mathcal{D}_T(C(K))} \sqrt{2}G$, where $G = \{(G_t(x);\ x \in E,\ t \geq 0\}$ is a centered Gaussian process with the following covariance function:*

$$\mathbb{E}\{G_t(x)G_s(y)\} = (s \wedge t)\ g(x,y), \qquad s,t \geq 0,\ x,y \in E.$$

Remark 2.2 Note that for each fixed x, $t \mapsto G_t(x)$ is a Brownian motion with infinitesimal variance $g(x,x)$. The latter is finite, due to the existence of local times; cf. GETOOR AND KESTEN (1972), for instance. On the other hand, for each fixed t, $x \mapsto G_t(x)$ has the same finite dimensional distributions as the Gaussian process $x \mapsto t^{-\frac{1}{2}}\eta_x$, where η is the Gaussian process associated to X. □

Remark 2.3 It is not hard to show that G has a continuous modification. Indeed, according the previous remark, $x \mapsto G_t(x)$ is continuous a.s. for each $t \geq 0$. Thus, by the general theory of Gaussian processes, this alone implies that $\mathbb{E}\{\sup_{x\in K} |G_t(x)|^p\} < +\infty$ for any compact $K \subset E$, and for all $p > 0$; cf. Borell's inequality in Adler (1990, Th. 2.1).

By Remark 2.2, $t \mapsto G_t$ is Brownian motion. Combined with the preceeding paragraph, this shows that $t \mapsto G_t$ is a Brownian motion on the space of continuous function on E. In particular, standard estimates show that for $s,t \geq 0$, and for all compact sets K in E,

$$\mathbb{E}\{\sup_{x\in K} |G_t(x) - G_s(x)|^p\} \leq C_{p,K}|t-s|^{\frac{p}{2}},$$

where $p > 2$ and $C_{p,K}$ is a finite constant that depends on K and p only. By Kolmogorov's continuity theorem, the asserted continuity of G follows. See REVUZ AND YOR (1991, Th. 2.1) for an appropriate version of Kolmogorov's theorem. □

Before proving Theorem 2.1, we mention, without proof, a lemma on Lévy processes that is both elementary and well-known.

Lemma 2.4 *Suppose that for each* $0 \le \lambda \le +\infty$, $Z_\lambda = \{Z_\lambda(t);\ t \ge 0\}$ *is a Lévy process on* $\mathcal{D}_T(C(K))$, *where* $T > 0$ *and* K *is a compact subset of* E. *If for all* $t \ge 0$, $Z_\lambda(t) \stackrel{C(K)}{\longrightarrow} Z_\infty(t)$, *as* $\lambda \to +\infty$, *then, as* $\lambda \to +\infty$, *the finite dimensional distributions of* Z_λ *converge to those of* Z_∞.

We now prove Theorem 2.1.

Proof of Theorem 2.1 By Eq. (2.2), for any $t \ge 0$,

$$\big\{L^x_{\tau(\lambda t)} + \tfrac{1}{2}\eta_x^2;\ x \in E\big\} \stackrel{(d)}{=} \big\{\tfrac{1}{2}(\eta_x + \sqrt{2\lambda t})^2;\ x \in E\big\},$$

where $\stackrel{(d)}{=}$ denotes the equality of finite-dimensional distributions. Hence,

$$\Big\{Y_\lambda(x,t) + \frac{\eta_x^2}{2\lambda^{\frac{1}{2}}};\ x \in E\Big\} \stackrel{(d)}{=} \Big\{\frac{\eta_x}{2\lambda^{\frac{1}{2}}}\big(\eta_x + 2\sqrt{2\lambda t}\big);\ x \in E\Big\}. \tag{2.4}$$

Clearly, as $\lambda \to +\infty$, $(\eta, 2 + \lambda^{-\frac{1}{2}}\eta)$ converges weakly in $C(K) \times C(K)$ to $(\eta, 2)$. Consequently, for any fixed $t \ge 0$, as $\lambda \to +\infty$,

$$Y_\lambda(\bullet, t) \stackrel{C(K)}{\longrightarrow} \sqrt{2t}\,\eta_\bullet.$$

By Remark 2.2, for each $t \ge 0$

$$Y_\lambda(\bullet, t) \stackrel{C(K)}{\longrightarrow} \sqrt{2}\,G_t(\bullet). \tag{2.5}$$

Furthermore, for every fixed $\lambda > 0$, $\{L^\bullet_{\tau(\lambda t)};\ t \ge 0\}$ is a Lévy process and for all $t \ge 0$, $K \ni x \mapsto L^x_{\tau(\lambda t)}$ is in $C(K)$. Consequently, by Lemma 2.4, we know that the finite dimensional distributions of $\{Y_\lambda(\bullet, t); t \ge 0\}$ converge to those of $\{\sqrt{2}G_t(\bullet); t \ge 0\}$.

In view of Eq. (2.8) and Ethier and Kurtz (1986, Th. 2.5, p. 167, Ch. 4), $\{Y_\lambda(\bullet, t); t \ge 0\}$ converges in law to a Markov process. The previous remark shows that this Markov process has the same law as $\{\sqrt{2}G_t(\bullet); t \ge 0\}$, from which our theorem follows. □

3 Proof of Theorem 1.3

Henceforth, η denotes the Gaussian process associated to X. Our proof of Theorem 1.3 rests on the following technical result.

Proposition 3.1 *Suppose* η *has a deterministic envelope at* $\mathbf{o} \in E$. *Then, for all compact* $K \subset E$ *that contain* $\mathbf{o}$,

$$\mathbb{P}_{\mathbf{o}}\{\exists t > 0 :\ \sup_{x \in K} G_t(x) \le 0\} = 0.$$

Proof We borrow an idea of Walsh (1986, p. 280) for this proof. Throughout, $\mathcal{F} = \{\mathcal{F}_t;\ t \ge 0\}$ denotes the right continuous augmented filtration of the infinite-dimensional, $C(K)$-valued process $t \mapsto G_t$, where the x-variable is restricted to some fixed compact set $K \subset E$ that includes the point $\mathbf{o}$.

By Remark 2.2, $t^{-\frac{1}{2}}G$ (restricted to K) has the same law as η (also restricted to K) for every $t > 0$. Consequently, Fubini's theorem and the existence of an envelope, together imply the existence of a sequence $\{x_n\}_{n\geq 1}$ converging to $\mathbf{o}$, and a function φ with $\varphi(\mathbf{o}) = 0 < \varphi(x_n)$, such that with probability one,

$$\limsup_{n\to\infty} \frac{G_t(x_n)}{\varphi(x_n)} = \sqrt{t}, \qquad \text{for almost every } t > 0. \tag{3.1}$$

Suppose that there exists a nonrandom $\delta \in (0,1)$, such that with positive probability,

$$\exists t > 0: \qquad \limsup_{n\to\infty} \frac{G_t(x_n)}{\varphi(x_n)} \leq (1-\delta)\sqrt{t}. \tag{3.2}$$

Thanks to progressive measurability, and thanks to the section theorem, for any $\varepsilon > 0$, we can find an $\mathcal{F}$-stopping time τ, such that (i) with probability at least ε, $\{0 < \tau < +\infty\}$ agrees with the collection of all ω for which (3.2) holds; and (ii) on $\{0 < \tau < \infty\}$,

$$\limsup_{n\to\infty} \frac{G_\tau(x_n)}{\varphi(x_n)} \leq (1-\delta)\sqrt{\tau};$$

see Dellacherie and Meyer (1975, Th. 44) for the section theorem. Our goal is to show that

$$\mathbb{P}_{\mathbf{o}}\{0 < \tau < +\infty\} = 0. \tag{3.3}$$

Since $\varepsilon > 0$ is arbitrary, this would prove our theorem.

By the strong Markov property, on $\{0 < \tau < \infty\}$, $t \mapsto G_{\tau+t} - G_\tau$ is a copy of G that is also independent of $\mathcal{F}_\tau$. Thus, on $\{0 < \tau < \infty\}$,

$$\limsup_{n\to\infty} \frac{G_{\tau+t}(x_n)}{\varphi(x_n)} \leq (1-\delta)\sqrt{\tau} + \limsup_{n\to\infty} \frac{G_{\tau+t}(x_n) - G_\tau(x_n)}{\varphi(x_n)}.$$

Hence, thanks to (3.1), on $\{0 < \tau < \infty\}$,

$$\limsup_{n\to\infty} \frac{G_{\tau+t}(x_n)}{\varphi(x_n)} \leq (1-\delta)\sqrt{\tau} + \sqrt{t}, \qquad \text{for almost every } t > 0.$$

On the other hand, note the real-variable inequality:

$$\left(1 - \tfrac{\delta}{2}\right)\sqrt{t+\tau} \geq (1-\delta)\sqrt{\tau} + \sqrt{t}, \qquad \forall 0 < t \leq \left(\tfrac{\delta}{2}\right)^2 \tau.$$

In particular, on $\{0 < \tau < \infty\}$,

$$\limsup_{n\to\infty} \frac{G_{\tau+t}(x_n)}{\varphi(x_n)} \leq \left(1 - \tfrac{\delta}{2}\right)\sqrt{t+\tau}, \qquad \text{on a } t\text{-set of positive Lebesgue measure.}$$

Since $\delta \in (0,1)$, this would contradict (3.1) unless Eq. (3.3) holds. □

We have established the requisite results for our proof of Theorem 1.3.

Proof of Theorem 1.3 Fix $\mu, T > 0$ and a compact $K \subset E$, and consider the measurable set

$$\begin{aligned}
\mathsf{C}_\mu &= \left\{\omega \in \mathcal{D}_T(C(K)) : \exists t \in [0,T]: \ \omega \in \left[Q_1(t) \cup Q_2(t)\right] \cap \left[Q_3(t) \cup Q_4(t)\right]\right\}, \text{ where} \\
Q_1(t) &= \left\{\omega \in \mathcal{D}_T(C(K)) : \ \sup_{x\in K} \omega(x,t) \leq 0\right\} \\
Q_2(t) &= \left\{\omega \in \mathcal{D}_T(C(K)) : \ \sup_{x\in K} \omega(x,t-) \leq 0\right\} \\
Q_3(t) &= \left\{\omega \in \mathcal{D}_T(C(K)) : \ \inf_{x\in K} \omega(x,t) \leq -\mu\right\} \\
Q_4(t) &= \left\{\omega \in \mathcal{D}_T(C(K)) : \ \inf_{x\in K} \omega(x,t-) \leq -\mu\right\}.
\end{aligned}$$

Since $\mathsf{C}_\mu(t)$ is closed in $\mathcal{D}_T(C(K))$, thanks to Theorem 2.1, and by properties of weak convergence,

$$\limsup_{\lambda\to\infty} \mathbb{P}_{\mathbf{o}}\big\{Y_\lambda \in \mathsf{C}_\mu\big\} \le \mathbb{P}_{\mathbf{o}}\big\{\sqrt{2}\,G \in \mathsf{C}_\mu\big\}$$
$$\le \mathbb{P}_{\mathbf{o}}\big\{\exists t \in [0,T]:\ \sqrt{2}\,G_t \in Q_1(t)\cap Q_3(t)\big\},$$

since G is continuous; cf. Remark 2.3. By Proposition 3.1,

$$\mathbb{P}_{\mathbf{o}}\big\{\exists t \in (0,T]:\ \sqrt{2}\,G_t \in Q_1(t)\cap Q_3(t)\big\} = 0.$$

On the other hand, since $G_0(\bullet)\equiv 0$, and since $\mu>0$,

$$\mathbb{P}_{\mathbf{o}}\big\{\sqrt{2}\,G_0 \in Q_3(0)\big\} = 0.$$

Consequently, we have shown that

$$\lim_{\lambda\to\infty} \mathbb{P}_{\mathbf{o}}\big\{Y_\lambda \in \mathsf{C}_\mu\big\} = 0. \tag{3.4}$$

Recalling Eq. (1.2), define

$$\mathcal{V}_t^K = \big\{x \in E:\ L_t^x = \sup_{y\in K} L_t^y\big\}.$$

Then, thanks to Eq. (3.4),

$$\lim_{\lambda\to\infty} \mathbb{P}_{\mathbf{o}}\left\{\begin{array}{c}\exists t \in [0,T]:\ \big[\mathbf{o}\in\mathcal{V}^K_{\tau(\lambda t)} \text{ or } \mathbf{o}\in\mathcal{V}^K_{\tau(\lambda t-)}\big] \text{ and} \\ \big[\inf_{x\in K} Y_\lambda(x,t) \le -\mu \text{ or } \inf_{x\in K} Y_\lambda(x,t-) \le -\mu\big]\end{array}\right\} = 0.$$

Since $Y_\lambda(x,0)=0$ for all $x\in E$, and since $\mu>0$, this shows that

$$\lim_{\lambda\to\infty} \mathbb{P}_{\mathbf{o}}\left\{\begin{array}{c}\exists t \in (0,T]:\ \big[\mathbf{o}\in\mathcal{V}^K_{\tau(\lambda t)} \text{ or } \mathbf{o}\in\mathcal{V}^K_{\tau(\lambda t-)}\big] \text{ and} \\ \big[\inf_{x\in K} Y_\lambda(x,t) \le -\mu \text{ or } \inf_{x\in K} Y_\lambda(x,t-) \le -\mu\big]\end{array}\right\} = 0. \tag{3.5}$$

The difference between this and the previous display is in the fact that the closed interval $[0,T]$ is now replaced with the half-open interval $(0,T]$. Next, consider the set $\mathsf{B}_\mu \subset D_T(C(K))$, defined by

$$\mathsf{B}_\mu = \big\{\omega \in \mathcal{D}_T(C(K)):\ \forall t\in(0,T],\ \inf_{x\in K}\omega(x,t) \le -\mu \text{ or } \inf_{x\in K}\omega(x,t-) \le -\mu\big\}. \tag{3.6}$$

We propose to show that

$$\lim_{\mu\to 0}\liminf_{\lambda\to\infty} \mathbb{P}_{\mathbf{o}}\{Y_\lambda \in \mathsf{B}_\mu\} = 1. \tag{3.7}$$

This would imply the theorem as we argue next. Indeed, Eq.'s (3.5) and (3.7) combine to show that

$$\lim_{\lambda\to\infty} \mathbb{P}_{\mathbf{o}}\big\{\exists t\in(0,T]:\ \mathbf{o}\in\mathcal{V}^K_{\tau(\lambda t)} \text{ or } \mathbf{o}\in\mathcal{V}^K_{\tau(\lambda t-)}\big\} = 0.$$

As a result,

$$\mathbb{P}_{\mathbf{o}}\big\{\exists r>0:\ L^{\mathbf{o}}_{\tau(r)} = \sup_{x\in K} L^x_{\tau(r)} \text{ or } L^{\mathbf{o}}_{\tau(r-)} = \sup_{x\in K} L^x_{\tau(r-)}\big\} = 0. \tag{3.8}$$

Now, let us suppose, to the contrary, that there exists a (random) $\mathfrak{t}>0$ such that $\sup_{x\in K} L^x_{\mathfrak{t}} = L^{\mathbf{o}}_{\mathfrak{t}}$, and set $L^{\mathbf{o}}_{\mathfrak{t}} = \ell$. Then, by the continuity of local times and using the inequality $\tau(\ell-)\le\mathfrak{t}\le\tau(\ell)$, we obtain

$$\ell \le \sup_{x\in K} L^{\mathbf{o}}_{\tau(\ell-)} \le \sup_{x\in K} L^x_{\mathfrak{t}} = \ell.$$

In particular, we would have $\ell = \sup_{x\in K} L^x_{\tau(\ell-)}$, which would contradict Eq. (3.8) unless

$$\mathbb{P}_{\mathbf{o}}\{\exists t > 0 : \; L^{\mathbf{o}}_t = \sup_{x\in K} L^x_t\} = 0. \tag{3.9}$$

Since E is assumed to be σ-compact, we can choose compact sets $K_1 \subset K_2 \subset \cdots$ that exhaust E. On the other hand, by the right-continuity of $t \mapsto X_t$, for every $m > 0$,

$$\lim_{n\to\infty} \mathbb{P}_{\mathbf{o}}\{\exists t \in [0,m] : \; \sup_{x\in K_n} L^x_t < \sup_{y\in E} L^y_t\} = 0.$$

In conjunction with Eq. (3.9), this easily yields our theorem. Thus, it suffices to demonstrate (3.7).

For any $\mu > 0$,

$$\begin{aligned} \mathsf{B}_\mu^{\complement} &\subseteq \big\{\omega \in \mathcal{D}_T(C(K)) : \exists t \in (0,T], \; \inf_{x\in K} \omega(x,t) > -\mu \text{ and } \inf_{x\in K} \omega(x,t-) > -\mu\big\} \\ &\subseteq \big\{\omega \in \mathcal{D}_T(C(K)) : \exists t \in (0,T], \; \inf_{x\in K} \omega(x,t) \geq -2\mu \text{ and } \inf_{x\in K} \omega(x,t-) \geq -2\mu\big\} \\ &= \mathsf{A}_\mu. \end{aligned}$$

Since A_μ is closed, Theorem 2.1 implies that

$$\begin{aligned} \limsup_{\lambda\to\infty} \mathbb{P}_{\mathbf{o}}\{Y_\lambda \in \mathsf{B}_\mu\} &\leq \mathbb{P}_{\mathbf{o}}\{\sqrt{2}G \in \mathsf{A}_\mu\} \\ &= \mathbb{P}_{\mathbf{o}}\{\exists t \in (0,T] : \; \inf_{x\in K} G_t(x) \geq -\sqrt{2}\mu\}, \end{aligned}$$

since by Remark 2.3, G is continuous. In particular,

$$\limsup_{\mu\to 0} \limsup_{\lambda\to\infty} \mathbb{P}_{\mathbf{o}}\{Y_\lambda \in \mathsf{B}_\mu^{\complement}\} \leq \mathbb{P}_{\mathbf{o}}\{\exists t \in (0,T] : \; \inf_{x\in K} G_t(x) \geq 0\big\} = 0,$$

thanks to Proposition 3.1. This demonstrates (3.7) and completes our proof. □

4 Proof of Proposition 1.4

Proposition 1.4 is an immediate consequence of Theorem 1.3, and the following refinement/variant of Arcones (1995, Lemma 2.1).

Lemma 4.1 *Suppose $\xi_1, \xi_2, \ldots$ are jointly standard Gaussian variates, and assume that*

$$\lim_{k\to\infty} (\ln k)^{\frac{1}{2}} \sup_{\substack{n,m\in\mathbb{Z}_+: \\ |m-n|\geq k}} \big|\mathbb{E}\{\xi_n \xi_m\}\big| = 0. \tag{4.1}$$

Then, $\limsup_{n\to\infty} (2\ln n)^{-\frac{1}{2}} \xi_n = 1$, *a.s.*

Proof Throughout, we write for all $x \in \mathbb{R}$, $\bar{\Phi}(x) = \mathbb{P}\{\xi_1 > x\}$, and recall Mill's ratios, viz.,

$$\frac{1 - x^{-2}}{x\sqrt{2\pi}} e^{-x^2/2} \leq \bar{\Phi}(x) \leq \frac{1}{x\sqrt{2\pi}} e^{-x^2/2}, \qquad \forall x > 1. \tag{4.2}$$

see Shorack and Wellner (1986, p. 850), for the latter, for instance.

As usual, one proves such a result in two steps. The first step is completely standard. Indeed, by Eq. (4.2), for any $n, \vartheta > 1$, $\mathbb{P}\{\xi_n > \sqrt{2\vartheta \ln n}\} \le n^{-\vartheta}$, which sums in n. Thus, the Borel–Cantelli lemma shows us that a.s., $\limsup_n (2\ln n)^{-\frac{1}{2}} \xi_n \le \vartheta$, for any $\vartheta > 1$. Choose $\vartheta \downarrow 1$ along a rational sequence to deduce that $\limsup_n (2\ln n)^{-\frac{1}{2}} \xi_n \le 1$, a.s. The second half proves the converse inequality. This is where the correlation condition (4.1) enters the picture.

Choose $\vartheta \in (0,1)$, and note that, thanks to Eq. (4.2), there exist constants, c_1 and c_2, such that for all $N > 2$,

$$c_1 N^{1-\vartheta} (\ln N)^{-\frac{1}{2}} \le \sum_{n=1}^{N} \mathbb{P}\{\xi_n > \sqrt{2\vartheta \ln n}\} \le c_2 N^{1-\vartheta} (\ln N)^{-\frac{1}{2}}. \tag{4.3}$$

In particular, since $\vartheta \in (0,1)$, the above sum goes to infinity as $N \to \infty$. In light of the Paley–Zygmund inequality, it suffices to show that

$$\mathbb{E}\Big\{ \Big| \sum_{n=1}^{N} \mathbf{1}_{\{\xi_n > \sqrt{2\vartheta \ln n}\}} \Big|^2 \Big\} \le (1 + o(1)) \cdot \Big| \mathbb{E}\Big\{ \sum_{n=1}^{N} \mathbf{1}_{\{\xi_n > \sqrt{2\vartheta \ln n}\}} \Big\} \Big|^2, \qquad \text{as } N \to \infty. \tag{4.4}$$

We follow Khoshnevisan and Shi (2000) to estimate the above. Indeed, for any $a, b > 0$, and writing $\rho_{n,m} = \mathbb{E}\{\xi_n \xi_m\}$ for the correlation,

$$\begin{aligned}
\mathbb{P}\{\xi_n > a, \xi_m > b\} &= \frac{1}{2\pi\sqrt{1-\rho_{n,m}^2}} \int_a^\infty \int_b^\infty \exp\Big(- \frac{x^2 + y^2 - 2\rho_{n,m} xy}{2(1-\rho_{n,m}^2)} \Big)\, dx\, dy \\
&\le \frac{1}{2\pi\sqrt{1-\rho_{n,m}^2}} \int_a^\infty \int_b^\infty \exp\Big(- \frac{(1-4\rho_{n,m}^+)}{2(1-\rho_{n,m}^2)} (x^2+y^2) \Big)\, dx\, dy,
\end{aligned}$$

where $\rho_{n,m}^+ = \max(\rho_{n,m}, 0)$. A change of variables yields the following, as long as $\rho_{n,m}^+ < \frac{1}{4}$:

$$\mathbb{P}\{\xi_n > a, \xi_m > b\} \le \frac{\sqrt{1-\rho_{n,m}^2}}{1-4\rho_{n,m}^+}\, \bar{\Phi}\left(a \sqrt{\frac{1-4\rho_{n,m}^+}{1-\rho_{n,m}^2}} \right) \bar{\Phi}\left(b \sqrt{\frac{1-4\rho_{n,m}^+}{1-\rho_{n,m}^2}} \right). \tag{4.5}$$

Eq. (4.4) follows from Eq.'s (4.1), (4.3), and (4.5), and a few lengthy computations. We will omit the details, as they follow similar ideas used in the second moment calculations of Khoshnevisan and Shi (2000). □

5 Symmetric Lévy Processes

In this section, we verify the condition of Proposition 1.4 in case X is a symmetric Lévy process on $E = \mathbb{R}$. We will write 0 for the distinguished point $\mathbf{o}$, as it makes sense to do so, and assume that

$$\int_1^\infty \frac{d\xi}{\Psi(\xi)} < +\infty, \quad \text{whereas} \quad \int_0^1 \frac{d\xi}{\Psi(\xi)} = +\infty, \tag{5.1}$$

where Ψ is the Lévy exponent of X. That is, $\mathbb{E}\{e^{i\xi X_t}\} = e^{-t\Psi(\xi)}$. The convergence of the first integral in (5.1) is equivalent to the existence of local times (Kesten 1969, Th. 2), while the divergence of the second integral is equivalent to recurrence (Port and Stone 1971, Th. 16.2). We also assume that the local times are continuous; see (Barlow 1988; Marcus and Rosen

1992) for an analytical condition in terms of Ψ that is equivalent to the mentioned continuity of local times. We recall the *potential kernel* for the recurrent process X is defined as

$$a(x) = \frac{1}{\pi} \int_0^\infty \frac{1 - \cos(x\lambda)}{\Psi(\lambda)}\, d\lambda, \qquad \forall x \in \mathbb{R}; \tag{5.2}$$

cf. (Bertoin 1996; Sato 1999). Then, it is well-known that

$$g(x, y) = a(x) + a(y) - a(x - y), \qquad \forall x, y \in \mathbb{R}. \tag{5.3}$$

This identity can be found in (Barlow 1988; Eisenbaum et al 1999; Getoor and Kesten 1972); see also (Bertoin 1996; Sato 1999) for pedagogical treatments.

5.1 Symmetric Stable Processes

First, consider X to be a symmetric stable process on $E = \mathbb{R}$ with index $\alpha \in (1, 2]$. We note that Eq. (5.1) holding, so that X is both recurrent and has local times. Moreover, according to Boylan (1964), X has continuous local times. Thus, all conditions of Theorem 1.3 are verified. In this case, we have

Theorem 5.1 (Eisenbaum 1997) *If X is a symmetric stable process of index $\alpha \in (1, 2]$, any $x \in \mathbb{R}$ is polar for the most visited sites.*

This was shown by different arguments. We now show how this result follows from our Theorem 1.3 (via Proposition 1.4), and also use this opportunity to fill a small gap in the proof of Eisenbaum (1997).

Proof We begin by computing the potential a defined in (5.2). Recall that $\Psi(\lambda) = \chi|\lambda|^\alpha$ for some $\chi > 0$. Consequently,

$$a(x) = c_\alpha |x|^{\alpha - 1}, \tag{5.4}$$

where $c_\alpha = (\chi\pi)^{-1} \int_0^\infty \lambda^{-\alpha}[1 - \cos(\lambda)]\, d\lambda$. On the other hand,

$$\begin{aligned} |a(y) - a(x - y)| &\le \frac{1}{\pi\chi} \int_0^\infty \frac{|\cos(\lambda y) - \cos(\lambda(y - x))|}{\lambda^\alpha}\, d\lambda \\ &= \frac{y^{\alpha-1}}{\pi\chi} \int_0^\infty \frac{|\cos(\zeta) - \cos(\zeta(1 - \frac{x}{y}))|}{\zeta^\alpha}\, d\zeta \\ &\le \frac{2y^{\alpha-1}}{\pi\chi} \int_0^\infty \frac{\zeta\left(\frac{x}{y}\right) \wedge 1}{\zeta^\alpha}\, d\zeta. \end{aligned}$$

In the last step, we have used the inequality, $|\cos(w) - \cos(z)| \le 2\{|w - z| \wedge 1\}$, valid for all $w, z \in \mathbb{R}$. Consequently, we obtain

$$|a(y) - a(x - y)| \le C_\alpha y^{\alpha-1} \left(\frac{x}{y}\right)^{\alpha-1}, \tag{5.5}$$

where $C_\alpha = \frac{2}{\pi\chi}\{(2 - \alpha)^{-1} + (\alpha - 1)^{-1}\}$. We can combine Eq.'s (5.3), (5.4) and (5.5) to obtain the following: for all $0 < x < y < 1$,

$$\begin{aligned} \frac{g(x, y)}{[g(x, x)g(y, y)]^{\frac{1}{2}}} &\le \frac{1}{2}\left(\frac{a(x)}{a(y)}\right)^{\frac{1}{2}} + \frac{|a(y) - a(x - y)|}{2\sqrt{a(x)a(y)}} \\ &\le c'_\alpha \left(\frac{x}{y}\right)^{\frac{\alpha-1}{2}}, \end{aligned} \tag{5.6}$$

where $c'_\alpha = \frac{1}{2}(1 + C_\alpha^{-1})$. We apply this with $x = x_n$ and $y = x_m$, where $x_\ell = 2^{-\ell}$ ($\ell \geq 2$) to see that for all $k \geq 1$,

$$\sup_{\substack{m,n\in\mathbb{Z}_+:\\ |m-n|\geq k}} \frac{g(x_n, x_m)}{[g(x_n, x_m)g(x_m, x_m)]^{\frac{1}{2}}} \leq c'_\alpha 2^{-\frac{k(\alpha-1)}{2}}.$$

Since this is $o((\ln k)^{-\frac{1}{2}})$ as $k \to \infty$, the asserted result follows from Proposition 1.4. □

5.2 An Extension

Thanks to Theorem 1.3 and Proposition 1.4, Theorem 5.1 can be extended in various directions. We will outline one possibility next.

Suppose X has a Lévy exponent of form

$$\Psi(\xi) = |\xi|^\alpha f(|\xi|), \qquad \forall \xi \in \mathbb{R}, \tag{5.7}$$

where $\alpha \in (1, 2)$, $f : \mathbb{R}_+ \to (0, \infty)$ is a nondecreasing continuous function such that

$$\exists \varepsilon \in (0, 2-\alpha) \text{ such that } x \mapsto x^\varepsilon f\Big(\frac{1}{x}\Big) \text{ is also nondecreasing.} \tag{5.8}$$

It is not hard to check that Eq. (5.1) holds for such an X, viz.,

$$\int_1^\infty \frac{d\lambda}{\lambda^\alpha f(\lambda)} \leq \frac{1}{f(1)} \int_1^\infty \frac{d\lambda}{\lambda^\alpha} < +\infty.$$

On the other hand,

$$\int_0^1 \frac{d\lambda}{\lambda^\alpha f(\lambda)} \geq \frac{1}{f(1)} \int_0^1 \frac{d\lambda}{\lambda^\alpha} = +\infty.$$

Thus, Eq. (5.1) holds, as asserted. We then have

Theorem 5.2 *Suppose X is a symmetric Lévy process that satisfies the above conditions. Then, it has continuous local times, and any singleton is polar for $\mathcal{V}$.*

Proof The central assertion of this theorem is the polarity of singletons. Assuming that local times are continuous, for the time being, we will prove this assertion first. Continuity of local times is deferred to the end of this demonstration.

We begin with an estimate for the growth of the potential kernel a near 0; cf. (5.2). Throughout, we use the following representation that is obtained from (5.2) by a change of variables:

$$a(x) = \frac{x^{\alpha-1}}{\pi} \int_0^\infty \frac{1 - \cos(\zeta)}{\zeta^\alpha f\big(\frac{\zeta}{x}\big)}\, d\zeta. \tag{5.9}$$

Clearly, whenever $x \geq 0$,

$$a(x) \geq \frac{x^{\alpha-1}}{\pi} \int_0^1 \frac{1 - \cos(\zeta)}{\zeta^\alpha f\big(\frac{\zeta}{x}\big)}\, d\zeta \geq c_{1,\alpha} \frac{x^{\alpha-1}}{f\big(\frac{1}{x}\big)},$$

where $c_{1,\alpha} = \pi^{-1}\int_0^1 \zeta^{-\alpha}[1-\cos(\zeta)]\,d\zeta$. On the other hand, we can also write

$$a(x) = \frac{x^{\alpha-1}}{\pi}(J_1+J_2), \text{ where}$$
$$J_1 = \int_0^1 \frac{1-\cos(\zeta)}{\zeta^\alpha f\left(\frac{\zeta}{x}\right)}\,d\zeta, \qquad \text{and} \qquad J_2 = \int_1^\infty \frac{1-\cos(\zeta)}{\zeta^\alpha f\left(\frac{\zeta}{x}\right)}\,d\zeta.$$

We estimate J_1 and J_2 in reverse order. Since f is nondecreasing,

$$J_2 \le \frac{c_{2,\alpha}}{f\left(\frac{1}{x}\right)},$$

where $c_{2,\alpha} = \int_1^\infty \zeta^{-\alpha}\,d\zeta$. On the other hand, since $1-\cos(|\zeta|) \le \zeta^2$,

$$J_1 \le \int_0^1 \frac{\zeta^{2-\varepsilon-\alpha}}{x^{-\varepsilon} f_\varepsilon\left(\frac{x}{\zeta}\right)}\,d\zeta,$$

where $f_\varepsilon(r) = r^\varepsilon f(\frac{1}{r})$. Since f_ε is nondecreasing, whenever $\zeta \in (0,1)$, $f_\varepsilon(\frac{x}{\zeta}) \ge f_\varepsilon(x)$. Thus,

$$J_1 \le \frac{c_{3,\alpha}}{f\left(\frac{1}{x}\right)},$$

where $c_{3,\alpha} = \int_0^1 \zeta^{2-\varepsilon-\alpha}\,d\zeta$ is a finite constant since $0<\varepsilon<1<\alpha<2$. We summarize our efforts, thus far, as follows: for all $x \ge 0$,

$$C_{1,\alpha}\,\frac{x^{\alpha-1}}{f\left(\frac{1}{x}\right)} \le a(x) \le C_{2,\alpha}\,\frac{x^{\alpha-1}}{f\left(\frac{1}{x}\right)}, \tag{5.10}$$

where $C_{1,\alpha} = c_{1,\alpha}\pi^{-1}$, and $C_{2,\alpha} = \pi^{-1}[c_{2,\alpha}+c_{3,\alpha}]$. Next, we estimate $a(y)-a(y-x)$ for $0<x<y<1$. By (5.2),

$$\begin{aligned}|a(y)-a(x-y)| &\le \frac{1}{\pi}\int_0^\infty \frac{|\cos(\lambda y)-\cos(\lambda(y-x))|}{\lambda^\alpha f(\lambda)}\,d\lambda \\ &= \frac{y^{\alpha-1}}{\pi}\int_0^\infty \frac{|\cos(\zeta)-\cos(\zeta[1-\frac{x}{y}])}{\zeta^\alpha f\left(\frac{\zeta}{y}\right)}\,d\zeta.\end{aligned}$$

To this, we apply the inequality $|\cos(w)-\cos(z)| \le 2[|z-w|\wedge 1]$, valid for all $z,w\in\mathbb{R}$, and deduce

$$|a(y)-a(x-y)| \le \frac{2y^{\alpha-1}}{\pi}\int_0^\infty \frac{\left(\frac{x\zeta}{y}\right)\wedge 1}{\zeta^\alpha f\left(\frac{\zeta}{y}\right)}\,d\zeta = \frac{2y^{\alpha-1}}{\pi}[I_1+I_2], \tag{5.11}$$

where $I_1 = \int_0^{y/x}(\cdots)$ and $I_2 = \int_{y/x}^\infty(\cdots)$. Clearly,

$$\begin{aligned}I_1 &= \left(\frac{x}{y}\right)\int_0^{y/x}\frac{\zeta^{1-\alpha}\,d\zeta}{f\left(\frac{\zeta}{y}\right)} = \left(\frac{x}{y}\right)\int_0^{y/x}\frac{\zeta^{1-\alpha-\varepsilon}\,d\zeta}{y^{-\varepsilon}f_\varepsilon\left(\frac{y}{\zeta}\right)} \\ &\le \left(\frac{x}{y}\right)\int_0^{y/x}\frac{\zeta^{1-\alpha-\varepsilon}\,d\zeta}{y^{-\varepsilon}f_\varepsilon(x)} = \left(\frac{x}{y}\right)^{1-\varepsilon}\frac{1}{f\left(\frac{1}{x}\right)}\int_0^{y/x}\zeta^{1-\alpha-\varepsilon}\,d\zeta \\ &= \frac{1}{2-\alpha-\varepsilon}\left(\frac{x}{y}\right)^{\alpha-1}\frac{1}{f\left(\frac{1}{x}\right)}.\end{aligned} \tag{5.12}$$

On the other hand,

$$I_2 = \int_{y/x}^{\infty} \frac{d\zeta}{\zeta^\alpha f\left(\frac{\zeta}{y}\right)} \leq \frac{1}{f\left(\frac{1}{x}\right)} \int_{y/x}^{\infty} \zeta^{-\alpha}\, d\zeta$$
$$= \frac{1}{\alpha - 1}\left(\frac{x}{y}\right)^{\alpha-1} \frac{1}{f\left(\frac{1}{x}\right)}.$$

Combining this with Eq.'s (5.12) and (5.11), we obtain the following: for all $0 < x < y < 1$,

$$|a(y) - a(x-y)| \leq C_{3,\alpha} \frac{x^{\alpha-1}}{f\left(\frac{1}{x}\right)}, \tag{5.13}$$

where $C_{3,\alpha} = \frac{2}{\pi}\{(\alpha-1)^{-1} + (2-\alpha-\varepsilon)^{-1}\}$. Now, we verify the condition on Proposition 1.4 by estimating the correlation ratio: whenever $0 < x < y < 1$,

$$\frac{g(x,y)}{[g(x,x)g(y,y)]^{\frac{1}{2}}} \leq \frac{1}{2}\left(\frac{a(x)}{a(y)}\right)^{\frac{1}{2}} + \frac{|a(y) - a(y-x)|}{2[a(x)a(y)]^{\frac{1}{2}}} \qquad \text{(cf. Eq. (5.6))}$$
$$\leq C_{4,\alpha}\frac{\phi(x)}{\phi(y)}, \tag{5.14}$$

where $C_{4,\alpha} = \frac{1}{2}\{(C_{2,\alpha}/C_{1,\alpha})^{\frac{1}{2}} + (C_{3,\alpha}/C_{1,\alpha})\}$, and $\phi(x) = \{x^{\alpha-1}/f(\frac{1}{x})\}^{\frac{1}{2}}$. Now, ϕ is a nondecreasing continuous function, such that for all $x \in (0,1)$, $|\phi(x)|^2 \leq x^{\alpha-1}/f(1) \to 0$, as $x \to 0$. Hence, $\lim_{n\to\infty} x_n = 0$, where x_n is defined by $\phi(x_n) = 2^{-n}$, for all n sufficiently large, and otherwise chosen arbitrarily in $(0,1)$. Moreover, thanks to Eq. (5.14),

$$\sup_{\substack{m,n\in\mathbb{Z}_+:\\ |m-n|\geq k}} \frac{g(x_n, x_m)}{[g(x_n,x_n)g(x_m,x_m)]^{\frac{1}{2}}} \leq C_{4,\alpha}2^{-k} = o\big((\ln k)^{-\frac{1}{2}}\big), \qquad (\text{as } k \to \infty).$$

Proposition 1.4, then, shows that 0 is polar for the favorite points. The Markov property, and the latter fact, together imply the important part of Theorem 5.2. We now complete our argument by verifying that local times are continuous in this case.

We recall from (Barlow and Hawkes 1985; Marcus and Rosen 1992), that a sufficient condition for the continuity of local times is that $\int_{0+} \sqrt{\ln \mathcal{N}_d(r)}\, dr < +\infty$, where $\mathcal{N}_d$ is the metric entropy of a compact set (say $[0,1]$) in the pseudo-metric described by $d(x,y) = \sqrt{\mathbb{E}\{(\eta_x - \eta_y)^2\}}$. That is, $\mathcal{N}_d(r)$ denotes the smallest number of d-balls of radius $\leq r$, needed to cover $[0,1]$ (say). Simple computations reveal that $d(x,y) = g(x,x) + g(y,y) - 2g(x,y)$. Using Eq. (5.10), we can see that $d(x,y) = 2a(x-y)$. Consequently, by Eq. (5.10),

$$d(x,y) \leq 2C_{2,\alpha}|\phi(|x-y|)|^2, \qquad \forall x,y \in [0,1].$$

We have already seen that $|\phi(r)|^2 \leq r^{\alpha-1}/f(1)$ for all $r \in (0,1]$. This yields

$$d(x,y) \leq \frac{2C_{2,\alpha}}{f(1)}|x-y|^{\alpha-1}, \qquad \forall x,y \in [0,1].$$

This shows that for all $r \in (0,1]$, $\mathcal{N}_d(r) \leq \mathcal{M}(r)$, where the latter is the number of ordinary (Euclidean) intervals of length $\leq qr^{\frac{1}{\alpha-1}}$, needed to cover $[0,1]$, where $q = \{f(1)/2C_{2,\alpha}\}^{\frac{1}{\alpha-1}}$. As $r \to 0^+$, $\mathcal{M}(r) \sim q^{-1}r^{-\frac{1}{\alpha-1}}$, which shows the existence of some constant K, such that

$$\int_0^{\frac{1}{2}} \sqrt{\ln \mathcal{N}_d}(r)\, dr \leq \int_0^{\frac{1}{2}} \sqrt{\ln \mathcal{M}(r)}\, dr \leq K \int_0^{\frac{1}{2}} |\ln r|^{\frac{1}{2}}\, dr < +\infty,$$

as was claimed. $\square$

References

[1] Adler, R.J. (1990). *An Introduction to Continuity, Extrema, and Related Topics for General Gaussian Processes*, Institute of Mathematical Statistics, Lecture Notes-Monograph Series, Volume 12, Hayward, California

[2] Arcones, M.A. (1995). On the law of the iterated logarithm for Gaussian processes. *J. Theoret. Probab.*, **8**(4), 877–903.

[3] Barlow, M.T. (1988). Necessary and sufficient conditions for the continuity of local times of Lévy processes, *Ann. Prob.*, **16**(1), 1389–1427.

[4] Barlow, M.T. and J. Hawkes (1985). Applications de l'entropie métrique à la continuité des temps locaux ds processus de Lévy. *C. R. Acad. Sci. Paris*, Ser. A-B 301, **116**, 237-239.

[5] Bass, R.F., N. Eisenbaum and Z. Shi (2000). The most visited sites of symmetric stable processes. *Probab. Theory and Relat. Fields*, **116**, 391–404.

[6] Bass, R.F. and P.S. Griffin (1985). The most visited site of Brownian motion and simple random walk. *Z. Wahr. verw. Geb.*, **70**, 417–436.

[7] Bertoin, J. (1996). *Lévy processes.* Cambridge University Press.

[8] Blumenthal, R.M. and R.K. Getoor (1968). *Markov Processes and Potential Theory.* Academic Press, New York.

[9] Boylan, E.S. (1964). Local times for a class of Markov processes. *Illinois J. Math.*, **8**, 19–39.

[10] Dellacherie, C. and P.A.Meyer (1975). *Probabilités et potentiels* (chp. I à IV). Publications de l'Institut de Maths. de l'Université de Strasbourg XV (Hermann).

[11] Eisenbaum, N. (2002). On the infinite divisibility of squared Gaussian processes. In preparation.

[12] Eisenbaum, N. (1997). On the most visited sites by a symmetric stable process. *Prob. Theory Rel. Fields*, **107**, 527–535.

[13] Eisenbaum, N., H. Kaspi, M.B. Marcus, J. Rosen and Z. Shi (2000). A Ray-Knight theorem for symmetric Markov processes. *Ann. Prob.*, **28**(4), 1781–1796.

[14] Erdős, P. and P. Révész (1984). On the favourite points of a random walk. *Math. Structures, Comp. Math., Math. Modelling*, **2**, 152–157, Sofia.

[15] Ethier, S.N. and T. Kurtz (1986). *Markov Processes. Characterization and Convergence.* John Wiley & Sons, Inc., New York.

[16] Getoor, R.K. and H. Kesten (1972). Continuity of local times for Markov processes. *Comp. Math.*, **24**, 277–303.

[17] Kesten, H. (1969). Hitting probabilities of single points for processes with stationary independent increments. *Memoirs of the Amer. Math. Soc.*, **93**, Providence.

[18] Khoshnevisan, D. and Z. Shi (2000). Fast sets and points for fractional Brownian motion. *Sém. de Probab. XXXIV*, Lecture Notes in Math. **1729**, 393–416. Springer, Berlin.

[19] Lifshits, M.A. and Z. Shi (2001). The escape rate of favourite sites of simple random walk and Brownian motion. Preprint.

[20] Marcus, M.B. (2000). The most visited sites of certain Lévy processes. To appear in *J. Theoret. Prob.*

[21] Marcus, M.B. and J. Rosen (1992). Sample path properties of the local times of strongly symmetric Markov processes via Gaussian processes. *Ann. Prob.*, **20**(4), 1603–1684.

[22] Port, S.C. and Stone, C.J. (1971). Infinitely divisible processes and their potential theory, II. *Ann. Inst. Fourier, Grenoble*, **21**(2), 157–275.

[23] Revuz, D. and M. Yor (1991). *Continuous Martingales and Brownian Motion*, third edition (1999). Springer, Berlin.

[24] Sato, K.-I. (1999). *Lévy Processes and Infinitely Divisible Distributions.* Cambridge University Press, Cambridge.

[25] Sharpe, M. (1988). *General Theory of Markov Processes.* Academic Press, Inc., Boston, MA.

[26] Shorack, G.R. and J.A. Wellner. (1986) *Empirical Processes with Applications to Statistics.* Wiley, New York.

[27] Shi, Z. and Tóth, B. (2000). Favourite sites of random walk. Period. Math. Hungar., **41**(1-2), 237-249.

[28] Walsh, J.B. (1986). An introduction to stochastic partial differential equations. In *École de'été de probabilités de Saint-Flour, XIV*-1984, Lec. Notes in Math. **1180**, 265–439. Springer, Berlin.

N. Eisenbaum. Laboratoire de Probabilités et Modéle Aléatoire, Université de Paris, VI, 4, Place Jussieu, 75252, Paris Cedex 05, France.
nae@ccr.jussieu.fr

D. Khoshnevisan. Department of Mathematics, University of Utah, 155 S. 1400 E. JWB 233, Salt Lake City, UT 85112–0900, U. S. A.
davar@math.utah.edu
http://www.math.utah.edu/~davar