EXPOSITIONES MATHEMATICAE

Expo. Math. 21 (2003): 97–114

www.urbanfischer.de/journals/expomath

Intersections of Brownian Motions*

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Abstract: This article presents a survey of the theory of the intersections of Brownian motion paths. Among other things, we present a truly elementary proof of a classical theorem of A. Dvoretzky, P. Erdős and S. Kakutani. This proof is motivated by old ideas of P. Lévy that were originally used to investigate the curve of planar Brownian motion.

Keywords: Path intersections, Brownian motion, additive Brownian motion.

1 Introduction

In this survey article we present an elementary and self-contained introduction to the theory of intersections of Brownian motion. Our starting point is the following classical theorem of A. Dvoretzky, P. Erdős and S. Kakutani:

Theorem 1.1 (Dvoretzky et al 1950) *Consider two independent Brownian motions $B := \{B(t);\, t \geq 0\}$ and $B' := \{B'(t);\, t \geq 0\}$, both on $\mathbb{R}^d$, and both starting from the origin. Then,*

$$\mathrm{P}\left\{B\left((0,\infty)\right) \cap B'\left((0,\infty)\right) \neq \varnothing\right\} > 0 \iff d \leq 3.$$

In plain terms, the above states that "*the trajectories of two independent Brownian motions can intersect if and only if the spatial dimension d is at most three.*"

The original proof of Dvoretzky et al (1950) can be abridged as follows: According to Kakutani (1944a), for any nonempty Borel measurable set $G \subset \mathbb{R}^d$,

$$\mathrm{P}\big\{B((0,\infty)) \cap G \neq \varnothing\big\} > 0 \iff \mathsf{Cap}_{(d-2)}(G) > 0,$$

where $\mathsf{Cap}_{(\beta)}(G)$ equals (i) the β-dimensional Newtonian capacity of G if $\beta > 0$; (ii) the logarithmic capacity of G if $\beta = 0$; and (iii) the number one if $\beta < 0$. See also Dvoretzky et al (1950, Lemma 2). [N.B.: For our purposes, it is not important to know what these

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*Research partially supported by grants from the NSF and NATO

MSC 2000 Subject Classification: Primary 60-02, 60J65; secondary 60G60

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0732-0869/03/21/02-097 $15.00/0

capacities are. However, the interested reader can find a pedagogic treatment in CARLESON 1983 and KHOSHNEVISAN 2002.] Consequently, by the independence between B and B' we have

$$\mathrm{P}\left\{B\left((0,\infty)\right)\cap B'\left((0,\infty)\right)\neq\varnothing\right\}>0 \iff \mathrm{E}\left[\mathsf{Cap}_{(d-2)}\left(B'\left((0,\infty)\right)\right)\right]>0.$$

Therefore, Theorem 1.1 is reduced to showing that the above expectation is positive if and only if $d \leq 3$; i.e., the range of Brownian motion is expected to have positive $(d-2)$-dimensional capacity if and only if $d \leq 3$. Further, more recent, connections between intersections of independent Markov processes and capacities can be found in references [2, 4–6, 10–13, 15–17, 19–26, 32–36, 39, 42–46, 48, 49, 51, 52]. Needless to say, there is an enormous literature on this topic, and we have only cited the references that are immediately relevant to the present paper.

AIZENMAN (1985) proposes an alternative derivation of Theorem 1.1. This derivation is based on the fact that the Laplace transform of the function $t \mapsto \mathrm{E}\{\mathfrak{m}_d(B(0,t)\cap B'(0,t))\}$ satisfies a manageable differential inequality, where $\mathfrak{m}_d$ denotes Lebesgue's measure on $\mathbb{R}^d$. See also ALBEVERIO AND ZHOU (1996), FELDER AND FRÖHLICH (1985), and LAWLER (1982).

While the preceding methods contain ideas that are both deep and powerful, they are not elementary. The goal of this article is to present, among other things, an elementary proof of Theorem 1.1. We will also present a second proof of Theorem 1.1 that provides more quantitative estimates on hitting probabilities.

Our first proof is very simple in that it is based solely on the essential properties of Brownian motion, i.e., scaling and the Markov property. Thus, one can introduce this proof even in a classroom setting. Our presentation is motivated by the analysis of LÉVY (1948, Théorème 53, p. 256), who showed that a single Brownian trajectory has positive d-dimensional Lebesgue measure if and only if $d = 1$; see Section 2 below, as well as LÉVY (1940). In the recent mathematical physics literature, this and its variants are called *renormalization group* methods. For a sampler see (AIZENMAN 1985; ALBEVERIO AND ZHOU 1996; FELDER AND FRÖHLICH 1985; LAWLER 1982) together with their combined references. Also, such methods have been recently introduced in MUELLER AND TRIBE (2001) in order to estimate various hitting probabilities for the solutions of an interesting class of stochastic PDEs.

Our second method is only moderately more complicated in the present context, but has the added advantage of being the backbone in the theory of multiparameter processes; cf. KHOSHNEVISAN (2002).

Throughout this article, $\mathfrak{m}_d$ represents the d-dimensional Lebesgue measure, and "$\iff$" means "if and only if."

We conclude this section by reviewing some well-known properties of a d-dimensional Brownian motion B; we tacitly refer to these properties throughout.

- *(The Gaussian property)* $B(0) = 0$, and for any $t > 0$, $B(t)$ is a random vector whose d coordinates are independent mean-zero normal random variables with variance t.
- *(Time-reversal)* Given any $T > 0$, the process $\{B(T) - B(T-t);\ 0 \leq t \leq T\}$ is standard Brownian motion run until time T.

- *(The Markov property)* Given any $T > 0$, the process $\{B(t+T) - B(t);\ t \geq 0\}$ is a standard Brownian motion that is totally independent of $\{B(s);\ 0 \leq s \leq T\}$.
- *(Scaling)* Given any $\alpha > 0$, the process $\{\alpha^{-\frac{1}{2}} B(\alpha t);\ t \geq 0\}$ is a standard Brownian motion.
- *(The reflection principle)* If $d = 1$ and $T > 0$, then $\sup_{0 \leq s \leq T} B(s)$ has the same distribution as $|B(T)|$. Consequently, for any $d \geq 1$, $\sup_{0 \leq s \leq T} |B(s)|$ has finite moments of all orders.

Acknowledgements: The final draft of this manuscript contains some improvements over the earlier versions thanks to the efforts of a number of people. I owe my deepest gratitude to: Jeff Steif, Yimin Xiao, and an anonymous referee who found a few subtle typographical errors, Jean-Pierre Kahane, Greg Lawler, and Yuval Peres who provided me with a number of further insights and references, and Robert Dalang who encouraged me to write up my viewpoint on this subject. I wish to thank *Ecole Polytechnique Fèdèrale de Lausanne* for their hospitality; much of this work was carried out during a visit there.

2 Hitting Points

In a nutshell, Theorem 1.1 is equivalent to

$$\mathrm{P}\left\{0 \in A\left((0,\infty)^2\right)\right\} > 0 \iff d \leq 3, \tag{2.1}$$

where A denotes the two-parameter *additive Brownian motion* defined by

$$A(s,t) = B(s) - B'(t), \qquad \forall s,t \geq 0. \tag{2.2}$$

In other words, Theorem 1.1 asserts that additive Brownian motion *hits* the origin if and only if $d \leq 3$. This turns out to be equivalent to the following:

$$\forall x \in \mathbb{R}^d, \qquad \mathrm{P}\left\{x \in A\left((0,\infty)^2\right)\right\} > 0 \iff d \leq 3. \tag{2.3}$$

On the other hand, by Fubini's theorem,

$$\mathrm{E}\left[\mathfrak{m}_d\left(A\left((0,\infty)^2\right)\right)\right] = \int_{\mathbb{R}^d} \mathrm{P}\left\{x \in A\left((0,\infty)^2\right)\right\} dx.$$

[We recall that $\mathfrak{m}_d$ denotes the d-dimensional Lebesgue measure.] Thus, a seemingly weaker version of Theorem 1.1 is

Proposition 2.1 (Khoshnevisan 1999, Th. 8.2) *If A denotes the two-parameter additive Brownian motion of (2.2), then*

$$\mathrm{E}\left[\mathfrak{m}_d\left(A\left((0,\infty)^2\right)\right)\right] > 0 \iff d \leq 3.$$

In fact, we will see that Proposition 2.1 is not hard to prove, and implies Theorem 1.1. Our proof will utilize an idea of LÉVY (1940). We first illustrate this idea in the simpler setting of (one-parameter) Brownian motion. The discussion of two-parameter additive Brownian motion will follow afterwards.

Theorem 2.2 (Lévy 1948, Th. 53, p. 256) *The range of d-dimensional Brownian motion has zero Lebesgue measure if and only if $d \geq 2$. Equivalently,*

$$\mathrm{E}\big[\mathfrak{m}_d\big(B((0,\infty))\big)\big] > 0 \iff d = 1.$$

Proof When $d = 1$, this is an elementary consequence of the almost sure continuity of $t \mapsto B(t)$, together with the fact that B is not a constant function, almost surely. Henceforth, we consider d-dimensional Brownian motion where $d \geq 2$, and strive to show that its range has zero Lebesgue measure.

First assuming that $\mathrm{E}[\mathfrak{m}_d(B((0,1)))]$ is finite, we show that it is zero. Once this is done, the stationarity of the increments of Brownian motion implies that for any $s > 0$, $\mathrm{E}[\mathfrak{m}_d(B([s,s+1]))] = 0$, which yields the desired result, for then we have $\mathrm{E}[\mathfrak{m}_d(B((0,\infty)))] \leq \sum_{s=0}^{\infty} \mathrm{E}[\mathfrak{m}_d(B([s,s+1]))] = 0$.

By Brownian scaling, the random set $B([0,2])$ has the same distribution as $\sqrt{2}B([0,1])$. Consequently, scaling properties of Lebesgue's measure tell us that $\mathfrak{m}_d(B([0,2]))$ has the same distribution as $2^{\frac{d}{2}}\mathfrak{m}_d(B([0,1]))$. Thus,

$$\mathrm{E}\big[\mathfrak{m}_d\big(B([0,2])\big)\big] = 2^{\frac{d}{2}}\mathrm{E}\big[\mathfrak{m}_d\big(B([0,1])\big)\big]. \tag{2.4}$$

On the other hand, $B([0,2]) = B([0,1]) \cup B([1,2])$, so that by the inclusion–exclusion principle,

$$\mathrm{E}\big[\mathfrak{m}_d\big(B([0,2])\big)\big] = \mathrm{E}\big[\mathfrak{m}_d(B([0,1]))\big] + \mathrm{E}\big[\mathfrak{m}_d\big(B([1,2])\big)\big] - \mathrm{E}\big[\mathfrak{m}_d\big(B([0,1]) \cap B([1,2])\big)\big].$$

But the Lebesgue measure of $B([1,2])$ is the same as that of $B([1,2]) - B(1)$ which—thanks to the Markov property—is independent of $\mathfrak{m}_d(B([0,1]))$ and has the same distribution as $\mathfrak{m}_d(B([0,1]))$.[1] This shows that the first two terms on the right-hand side of the above diplay are equal; i.e.,

$$\mathrm{E}\big[\mathfrak{m}_d\big(B([0,2])\big)\big] = 2\mathrm{E}\big[\mathfrak{m}_d(B([0,1]))\big] - \mathrm{E}\big[\mathfrak{m}_d\big(B([0,1]) \cap B([1,2])\big)\big].$$

By first conditioning on $B(1)$, and then appealing to the Markov property and time-reversal, we deduce that

$$\mathrm{E}\big[\mathfrak{m}_d\big(B([0,2])\big)\big] = 2\mathrm{E}\big[\mathfrak{m}_d(B([0,1]))\big] - \mathrm{E}\big[\mathfrak{m}_d\big(B([0,1]) \cap B'([0,1])\big)\big].$$

In particular, we can combine this with (2.4) and deduce that

$$(2 - 2^{\frac{d}{2}})\mathrm{E}\big[\mathfrak{m}_d\big(B([0,1])\big)\big] = \mathrm{E}\big[\mathfrak{m}_d\big(B([0,1]) \cap B'([0,1])\big)\big]. \tag{2.5}$$

[1]Here and throughout, whenever A is a set and x is a point, $A \pm x$ denotes the collection of all $a \pm x$, $a \in A$.

If we write $\varphi(x) = \mathrm{P}\{x \in B([0,1])\}$ for $x \in \mathbb{R}^d$, then we have $\mathrm{E}[\mathfrak{m}_d(B([0,1]))] = \int_{\mathbb{R}^d} \varphi(x)\,dx$ and

$$\mathrm{E}[\mathfrak{m}_d(B([0,1]) \cap B'([0,1]))] = \int_{\mathbb{R}^d} \mathrm{P}\left\{x \in B([0,1]), x \in B'([0,1])\right\} dx = \int_{\mathbb{R}^d} \varphi^2(x)\,dx. \quad (2.6)$$

Hence, (2.5) can be recast as

$$(2 - 2^{\frac{d}{2}}) \int_{\mathbb{R}^d} \varphi(x)\,dx = \int_{\mathbb{R}^d} \varphi^2(x)\,dx.$$

Since $d \geq 2$, the left-hand side is nonpositive. Therefore, $\varphi(x) = 0$ almost everywhere, which is the desired result.

To complete this proof, it suffices to show that $\mathrm{E}[\mathfrak{m}_d(B([0,1]))] < +\infty$, but this is a consequence of the well-known reflection principle of D. André that we will not reproduce here; cf. Revuz and Yor (1991, Proposition 3.7, p. 100) for details. □

Theorem 2.2 should be viewed as the one-parameter Brownian analogue of Proposition 2.1. Now, we use it to prove the one-parameter Brownian analogue of Theorem 1.1. In words, the following states that points ($d \geq 2$) are almost surely not *hit* by the Brownian curve.

Proposition 2.3 (Lévy 1948, Cor. 53, p. 257) *If $d \geq 2$ and $z \in \mathbb{R}^d$, then*

$$\mathrm{P}\{z \in B((0,\infty))\} = 0.$$

Proof We first invoke scaling and the stationary increments property of B to reduce the problem to showing that $\mathrm{P}\{z \in B([1,2])\} = 0$; we will verify the latter property next.

Recall from Theorem 2.2 that $\varphi(x) = \mathrm{P}\{x \in B([0,1])\} = 0$ for almost every $x \in \mathbb{R}^d$. In particular,

$$\mathrm{E}\left\{\varphi\left(z - B(1)\right)\right\} = \int_{\mathbb{R}^d} \varphi(x+z) \frac{e^{-\frac{1}{2}|x|^2}}{(2\pi)^{\frac{d}{2}}}\,dx = 0, \qquad \forall z \in \mathbb{R}^d.$$

Thanks to the stationarity of the increments of Brownian motion, we can also write $\varphi(x) = \mathrm{P}\{x \in B([1,2]) - B(1)\}$. Therefore, by the Markov property, the preceding display equals $\mathrm{P}\{z \in B([1,2])\}$ that was earlier shown to be zero. □

3 Proof of Theorem 1.1

We now turn to our proof of Theorem 1.1. It should be recognized that we need to consider only the cases $d = 3$ and $d = 4$. However, in order to identify the key elements of this proof, we start with the case $d \geq 5$. The cases $d = 4$ and $d < 4$ are handled separately following the next subsection.

3.1 Nonintersection for $d \geq 5$

We now prove the "$d > 4$" portion of Theorem 1.1.

Proposition 3.1 *If $d \geq 5$, then $B((0,\infty)) \cap B'((0,\infty)) = \emptyset$, almost surely.*

We recall the additive Brownian motion A from (2.2), and note that Proposition 3.1 is equivalent to the statement that $0 \notin A((0,\infty)^2)$, almost surely. Moreover, after appealing to Brownian scaling and the stationarity of Brownian motion, we need only show that $0 \notin A([1,2]^2)$, almost surely. We will demonstrate this in two easy steps.

Step 1. We first show that $\mathfrak{m}_d(A([0,1]^2)) = 0$ almost surely; cf. Proposition 2.1 in the case $d > 4$.

Proof Scaling considerations (as in Section 2) reveal that

$$\mathrm{E}\big[\mathfrak{m}_d\big(A([0,2]^2)\big)\big] = 2^{\frac{d}{2}}\mathrm{E}\big[\mathfrak{m}_d\big(A([0,1]^2)\big)\big].$$

On the other hand,

$$\begin{aligned}\mathrm{E}\big[\mathfrak{m}_d\big(A([0,2]^2)\big)\big] \leq\ & \mathrm{E}\big[\mathfrak{m}_d\big(A([0,1]^2)\big)\big] + \mathrm{E}\big[\mathfrak{m}_d\big(A([0,1]\times[1,2])\big)\big] + \mathrm{E}\big[\mathfrak{m}_d\big(A([1,2]\times[0,1])\big)\big] \\ & + \mathrm{E}\big[\mathfrak{m}_d\big(A([1,2]^2)\big)\big].\end{aligned}$$

By the Markov property, all four expectations on the right-hand side are equal (and finite, by the reflection principle). Thus, we combine the preceding two displays to infer that

$$2^{\frac{d}{2}}\mathrm{E}\big[\mathfrak{m}_d\big(A([0,1]^2)\big)\big] \leq 4\mathrm{E}\big[\mathfrak{m}_d\big(A([0,1]^2)\big)\big]. \tag{3.1}$$

Since $d \geq 5$, the above expectation must be zero as claimed. □

Step 2. We now use Step 1 to show that with probability one, $0 \notin A([1,2]^2)$.

Proof If $\psi(x) = \mathrm{P}\{x \in A([0,1]^2)\}$, Step 1 implies that $\psi(x) = 0$ for almost every $x \in \mathbb{R}^d$. Consequently, for all $z \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} \psi(x+z)\frac{e^{-\frac{1}{4}|x|^2}}{(4\pi)^{\frac{d}{2}}}\,dx = 0.$$

Since the density function of $A(1,1)$ is $(4\pi)^{-\frac{d}{2}}\exp(-\frac{1}{4}|x|^2)$, the integral in the preceding display equals $\mathrm{P}\{z \in A([1,2]^2)\}$, which must be zero. Somewhat more generally, this argument shows that for all $z \in \mathbb{R}^d$, and all $0 < a < b$ and $0 < a' < b'$, $\mathrm{P}\{z \in A([a,b]\times[a',b'])\} = 0$, which implies that $\mathrm{P}\{z \in A((0,\infty)^2)\} = 0$. Step 2, and hence the proposition, follows from considering the special case $z = 0$. □

3.2 Nonintersection for $d = 4$

We now turn our attention to the more difficult case $d = 4$, and propose to show the following, all the time using the notation of §3.1:

Proposition 3.2 *If $d = 4$, then $B((0, \infty)) \cap B'((0, \infty)) = \varnothing$, almost surely.*

Proof We only need to verify Proposition 2.1 in the case $d = 4$; i.e., we have to prove that $\mathrm{E}\{\mathfrak{m}_4(A((0, \infty)^2))\} = 0$. Once this is verified, the remainder of the proof follows Step 2 of our proof of Proposition 3.1 verbatim.

When $d = 4$, things are trickier than in the $d > 4$ case; this can be seen from (3.1) which is now a tautology. With this in mind, we aim to find the correction in (3.1). Indeed, note that

$$\mathrm{E}\big[\mathfrak{m}_4\big(A([0,2]^2)\big)\big] \le \mathrm{E}\big[\mathfrak{m}_4\big(A([0,1] \times [0,2])\big)\big] + \mathrm{E}\big[\mathfrak{m}_4\big(A([1,2] \times [0,2])\big)\big].$$

By the stationarity of the increments of B, the terms on the right-hand side are equal. Furthermore, since $d = 4$, scaling enforces the relation $\mathrm{E}[\mathfrak{m}_4(A([0,2]^2))] = 4\mathrm{E}\,[\mathfrak{m}_4(A([0,1]^2))]$. Combining terms, we obtain

$$4\mathrm{E}\big[\mathfrak{m}_4\big(A([0,1]^2)\big)\big] \le 2\mathrm{E}\big[\mathfrak{m}_4\big(A([0,1] \times [0,2])\big)\big].$$

On the other hand,

$$\begin{aligned}
&\mathrm{E}\big[\mathfrak{m}_4\big(A([0,1] \times [0,2])\big)\big] \\
&= \mathrm{E}\big[\mathfrak{m}_4\big(A([0,1]^2)\big)\big] + \mathrm{E}\big[\mathfrak{m}_4\big(A([0,1] \times [1,2])\big)\big] - \mathrm{E}\big[\mathfrak{m}_4\big(A([0,1]^2) \cap A([0,1] \times [1,2])\big)\big] \\
&= 2\mathrm{E}\big[\mathfrak{m}_4\big(A([0,1]^2)\big)\big] - \mathrm{E}\big[\mathfrak{m}_4\big(A([0,1]^2) \cap A([0,1] \times [1,2])\big)\big] \\
&= 2\mathrm{E}\big[\mathfrak{m}_4\big(A([0,1]^2)\big)\big] - \mathrm{E}\big[\mathfrak{m}_4\big([B([0,1]) - B'([0,1])] \cap [B([0,1]) - B'([1,2])]\big)\big].
\end{aligned}$$

The preceding two displays together imply that

$$\mathrm{E}\left[\mathfrak{m}_4\Big([B([0,1]) - B'([0,1])] \cap [B([0,1]) - B'([1,2])]\Big)\right] = 0.$$

Equivalently, the following holds with probability one:

$$\mathrm{E}\left[\mathfrak{m}_4\Big([B([0,1]) + B'([0,1])] \cap [B([0,1]) + B'([1,2])]\Big) \,\Big|\, B'(1)\right] = 0.$$

Thanks to the Markov property and time-reversal, given $\{B'(1) = w\}$, the random sets $B'([0,1])$ and $B'([1,2])$ are conditionally independent, each having the same distribution as the range of Brownian motion started at w. Thus, conditionally on $\{B'(1) = w\}$, the random set $[B([0,1]) + B'([0,1])] \cap [B([0,1]) + B'([1,2])]$ has the same distribution as $w + \{[B([0,1]) + B'([0,1])] \cap [B([0,1]) + B''([0,1])]\}$ where B'' is a third independent Brownian motion. Since the said Lebesgue measure is independent of w,

$$\mathrm{E}\left[\mathfrak{m}_4\Big([B([0,1]) + B'([0,1])] \cap [B([0,1]) + B''([0,1])]\Big)\right] = 0.$$

On the other hand, given $B([0,1])$, the two sets $B([0,1])+B'([0,1])$ and $B([0,1])+B''([0,1])$ have the same conditional distributions. Thus, using a reasoning similar to the one that led to (2.6) now leads us to the following:

$$\mathrm{E}\left[\int_{\mathbb{R}^4}\left(\mathrm{P}\left\{x\in A([0,1]^2)\,\middle|\,B([0,1])\right\}\right)^2 dx\right]=0.$$

In particular, $\mathrm{P}\{x \in A([0,1]^2)\} = 0$ for almost every $x \in \mathbb{R}^4$. From this we can conclude that if $0 < a < b$ and $0 < a' < b'$, then $\mathrm{E}[\mathfrak{m}_4(A([a,b] \times [a',b']))] = 0$. Proposition 3.2 follows in the case $d = 4$, and this concludes our argument. □

It is worth pointing out that in this and the previous section, we have shown the following refinement of Propositions 3.1 and 3.2:

$$d \geq 4 \implies \forall z \in \mathbb{R}^d: \; \mathrm{P}\{z \in B((0,\infty)) - B'((0,\infty))\} = 0. \tag{3.2}$$

3.3 Fourier Analysis for $d \leq 3$

Used in conjunction, the results of Sections 3.1 and 3.2 together show the nonintersection half of Theorem 1.1. In this subsection, we verify the other half by using the Fourier-analytical method of KAHANE (1983, 1985). Throughout, we use the notation of §3.1, and strive to prove the following result; the $d \leq 3$ portion of Theorem 1.1 is a ready consequence.

Theorem 3.3 (Dvoretzky et al 1950) *If $d \leq 3$, then $B((0,\infty)) - B'((0,\infty))$ has positive Lebesgue measure almost surely.*

Proof Recall the additive Brownian motion of (2.2), and consider its (weighted, or discounted) *occupation measure* ν defined, for all Borel sets $F \subset \mathbb{R}^d$, by

$$\nu(F) = \int_0^\infty \int_0^\infty e^{-s-t}\mathbf{1}_F(A(s,t))\,ds\,dt.$$

This is a random measure on the Borel subsets of $\mathbb{R}^d$ whose Fourier transform is

$$\widehat{\nu}(\xi) = \int_0^\infty \int_0^\infty e^{-s-t}e^{i\xi\cdot A(s,t)}\,ds\,dt, \qquad \forall \xi \in \mathbb{R}^d.$$

Our Fourier transform is scaled so that $\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{i\xi\cdot x}f(x)\,dx$ ($\xi \in \mathbb{R}^d$). Evidently, for all $\xi \in \mathbb{R}^d$,

$$\begin{aligned}
\mathrm{E}\left\{\left|\widehat{\nu}(\xi)\right|^2\right\} &= \iint\limits_{(0,\infty)^2}\iint\limits_{(0,\infty)^2} e^{-s-t-s'-t'}\mathrm{E}\left\{e^{i\xi\cdot A(s,t)-i\xi\cdot A(s',t')}\right\}\,ds\,dt\,ds'\,dt' \\
&= \iint\limits_{(0,\infty)^2}\iint\limits_{(0,\infty)^2} e^{-s-t-s'-t'}\mathrm{E}\left\{e^{i\xi\cdot B(s)-i\xi\cdot B(s')}\right\}\times\mathrm{E}\left\{e^{-i\xi\cdot B'(t)+i\xi\cdot B'(t')}\right\}\,ds\,dt\,ds'\,dt' \\
&= \left(\int_0^\infty\int_0^\infty \exp\left\{-s-s'-\frac{|\xi|^2}{2}|s-s'|\right\}\,ds\,ds'\right)^2 \\
&= \left(1+\frac{|\xi|^2}{2}\right)^{-2}.
\end{aligned}$$

Integrating $[d\xi]$, we obtain the following:

$$\mathrm{E}\left[\int_{\mathbb{R}^d} \left|\widehat{\nu}(\xi)\right|^2 d\xi\right] < +\infty \iff d \leq 3.$$

Thus, if $d \leq 3$, then $\widehat{\nu} \in \mathcal{L}^2(\mathfrak{m}_d)$ almost surely. In this case, by Plancherel's theorem, ν is absolutely continuous almost surely, which is more than what we need. □

Remark 3.4 The Radon-Nikodým derivative $\left(\frac{d\nu}{dx}\right)$ in the above proof is the all-important *local time* process of A; this is also known as the *intersection local time* for the processes B and B'. See GEMAN AND HOROWITZ (1980) for a general $\mathcal{L}^2$-theory for the local times of random fields. □

4 The Hitting Probability of a Small Ball

We return to our discussion of §2 to explain the second idea of this paper. Here, we will be looking more closely at the hitting probability of a small ball. While such estimates are quite classical (e.g., see KAKUTANI 1944b for $d = 3$), we will be interested in the following, more modern, formulation. Throughout, $\mathcal{B}_d(x, r)$ denotes the Euclidean ℓ^2-ball of radius $r > 0$ about $x \in \mathbb{R}^d$.

Theorem 4.1 *For any $0 < a < b$, there exists a finite constant $C = C(a, b, d) > 1$, such that*

$$C^{-1}\mathcal{K}_{d-2}(\varepsilon) \leq \mathrm{P}\{B([a,b]) \cap \mathcal{B}_d(0,\varepsilon) \neq \varnothing\} \leq C\mathcal{K}_{d-2}(\varepsilon), \qquad \forall \varepsilon \in (0,1),$$

where for all $\beta \in \mathbb{R}$,

$$\mathcal{K}_\beta(x) = \begin{cases} 1, & \text{if } \beta < 0, \\ \left[\log_+\left(\frac{1}{x}\right)\right]^{-1}, & \text{if } \beta = 0, \\ x^\beta, & \text{if } \beta > 0. \end{cases} \tag{4.1}$$

Remark 4.2 Our arguments can be refined to show that the upper bound holds when the probability in question is replaced by the larger quantity $\sup_{z \in \mathbb{R}^d} \mathrm{P}\{B([a,b]) \cap \mathcal{B}_d(z,\varepsilon) \neq \varnothing\}$. □

For a proof, we first need a classical inequality of Paley and Zygmund that has been independently rediscovered many times since its original discovery.

Lemma 4.3 (Paley and Zygmund 1932) *If Z is a positive random variable, then*

$$\mathrm{P}\{Z > 0\} \geq \frac{\left(\mathrm{E}[Z]\right)^2}{\mathrm{E}[Z^2]},$$

provided that the expectations exist and are strictly positive.

Proof By the Cauchy–Schwarz inequality,

$$\mathrm{E}[Z] = \mathrm{E}\left[Z\mathbf{1}_{(0,\infty)}(Z)\right] \leq \sqrt{\mathrm{E}[Z^2]\cdot \mathrm{P}\{Z>0\}},$$

where $\mathbf{1}_F(x) = 1$ if $x \in F$, and $\mathbf{1}_F(x) = 0$ if $x \in F^{\complement}$. The lemma follows. □

We will use the above to obtain the following, from which Theorem 4.1 follows after a few elementary computations that only involve the standard Gaussian density function.

Proposition 4.4 (Khoshnevisan 1997) *For all* $0 < a < b$, *and all* $\varepsilon > 0$,

$$\frac{\int_a^b \mathrm{P}\{|B(s)| \leq \varepsilon\}\, ds}{2\int_0^b \mathrm{P}\{|B(s)| \leq \varepsilon\}\, ds} \leq \mathrm{P}\{B([a,b]) \cap \mathcal{B}_d(0,\varepsilon) \neq \varnothing\} \leq \frac{\int_a^{2b} \mathrm{P}\{|B(s)| \leq 2\varepsilon\}\, ds}{\int_0^b \mathrm{P}\{|B(s)| \leq \varepsilon\}\, ds}.$$

Proof For any $0 < a < b$ and all $\varepsilon > 0$ define

$$J_{a,b}(\varepsilon) = \int_a^b \mathbf{1}_{\mathcal{B}_d(0,\varepsilon)}(B(s))\, ds. \tag{4.2}$$

Evidently, $\mathrm{E}[J_{a,b}(\varepsilon)] = \int_a^b \mathrm{P}\{|B(s)| \leq \varepsilon\}\, ds$. Furthermore,

$$\begin{aligned}\mathrm{E}\left[J_{a,b}^2(\varepsilon)\right] &= 2\int_a^b \int_s^b \mathrm{P}\{|B(s)| \leq \varepsilon\,,\ |B(t)| \leq \varepsilon\}\, dt\, ds \\ &= 2\int_a^b \int_s^b \mathrm{P}\{|B(s)| \leq \varepsilon\,,\ |B(t) - B(s) + B(s)| \leq \varepsilon\}\, dt\, ds.\end{aligned}$$

Since $B(t) - B(s)$ and $B(s)$ are independent,

$$\mathrm{E}\left[J_{a,b}^2(\varepsilon)\right] \leq 2\int_a^b \int_s^b \mathrm{P}\{|B(s)| \leq \varepsilon\} \times \sup_{z\in\mathbb{R}^d} \mathrm{P}\{|B(t) - B(s) + z| \leq \varepsilon\}\, dt\, ds.$$

The distribution of $B(t) - B(s)$, being centered and Gaussian, is unimodal. That is, the above "$\sup_{z\in\mathbb{R}^d}$" is achieved at $z = 0$. Consequently, we use the stationarity of the increments of B to deduce that

$$\mathrm{E}\left[J_{a,b}^2(\varepsilon)\right] \leq 2\int_a^b \mathrm{P}\{|B(s)| \leq \varepsilon\}\, ds \times \int_0^b \mathrm{P}\{|B(t)| \leq \varepsilon\}\, dt. \tag{4.3}$$

We can combine the Paley–Zygmund inequality (Lemma 4.3) with (4.3) to obtain

$$\mathrm{P}\{J_{a,b}(\varepsilon) > 0\} \geq \frac{\int_a^b \mathrm{P}\{|B(s)| \leq \varepsilon\}\, ds}{2\int_0^b \mathrm{P}\{|B(s)| \leq \varepsilon\}\, ds}.$$

This readily implies the asserted lower bound on the hitting probability of $\mathcal{B}_d(0,\varepsilon)$. We now work toward the upper bound.

Consider the martingale $M(t) = \mathrm{E}[J_{a,2b}(2\varepsilon) \,|\, \mathcal{F}(t)]$ $(t \geq 0)$, where $\mathcal{F}(t)$ is the sigma-field generated by the collection $\{B(s);\, s \leq t\}$. We note that for any $t \in [a, b]$, the following holds almost surely:

$$\begin{aligned} M(t) &\geq \int_t^{2b} \mathrm{P}\{|B(s)| \leq 2\varepsilon \,|\, \mathcal{F}(t)\}\, ds \times \mathbf{1}_{\mathcal{B}_d(0,\varepsilon)}(B(t)) \\ &\geq \int_t^{2b} \mathrm{P}\{|B(s) - B(t)| \leq \varepsilon \,|\, \mathcal{F}(t)\}\, ds \times \mathbf{1}_{\mathcal{B}_d(0,\varepsilon)}(B(t)), \end{aligned}$$

thanks to the triangle inequality and the $\mathcal{F}(t)$-measurability of $B(t)$. By the stationarity and the independence of Brownian increments, we obtain the following: For all $t \in [a, b]$, almost surely,

$$\begin{aligned} M(t) &\geq \int_0^{2b-t} \mathrm{P}\{|B(s)| \leq \varepsilon\}\, ds \times \mathbf{1}_{\mathcal{B}_d(0,\varepsilon)}(B(t)) \\ &\geq \int_0^{b} \mathrm{P}\{|B(s)| \leq \varepsilon\}\, ds \times \mathbf{1}_{\mathcal{B}_d(0,\varepsilon)}(B(t)). \end{aligned}$$

Since M is a martingale with respect to the Brownian filtration $\mathcal{F}$, it is continuous; cf. Revuz and Yor (1991, Theorem 3.4, p. 187). Thus, there exists one null set off of which the above holds simultaneously for all $t \in [a, b]$. Consequently, we can replace t by the stopping time, $\tau = \inf\{s \in [a, b] : |B(s)| \leq \varepsilon\}$ where $\inf \varnothing = \infty$. Since $\mathbf{1}_{\mathcal{B}_d(0,\varepsilon)}(B(\tau)) = \mathbf{1}\{B([a, b]) \cap \mathcal{B}_d(0, \varepsilon) \neq \varnothing\}$, and since M is a bounded (continuous) martingale, Doob's optional stopping theorem (Revuz and Yor 1991, Theorem 3.2, p. 65) yields

$$\mathrm{E}[J_{a,2b}(2\varepsilon)] = \mathrm{E}[M(\tau)] \geq \int_0^b \mathrm{P}\{|B(s)| \leq \varepsilon\}\, ds \times \mathrm{P}\left\{B([a, b]) \cap \mathcal{B}_d(0, \varepsilon) \neq \varnothing\right\}.$$

The proposition follows readily from this. □

5 Near-Intersection Probabilities

We conclude this paper by using some of the salient features of the argument of Section 2, in conjunction with ideas of Cairoli (1968), in order to prove the following which estimates the probability of near-intersection of two indepedent Brownian motions; see (Aizenman 1985; Albeverio and Zhou 1996; Pemantle et al. 1996) for related estimates, and Proposition 4.4 for a one-parameter Brownian version.

Theorem 5.1 *For all $0 < a < b$ and $0 < a' < b'$, there exists a finite constant $C_\star = C_\star(a, b, a', b', d) > 1$, such that*

$$C_\star^{-1} \mathcal{K}_{d-4}(\varepsilon) \leq \mathrm{P}\{A([a, b] \times [a', b']) \cap \mathcal{B}_d(0, \varepsilon) \neq \varnothing\} \leq C_\star \mathcal{K}_{d-4}(\varepsilon), \qquad \forall \varepsilon \in (0, 1).$$

Remark 5.2
(a) Clearly, Theorem 5.1 implies Theorem 1.1.
(b) The function $\mathcal{K}_{d-4}$ is defined in (4.1). □

Proof (sketch) Consider the two-parameter analogue of (4.2),

$$J_{a,b,a',b'}(\varepsilon) = \int_a^b \int_{a'}^{b'} \mathbf{1}_{\mathcal{B}_d(0,\varepsilon)}(A(s,s'))\, ds\, ds', \qquad \forall \varepsilon \in (0,1).$$

We can note the following lower bound on the probability density of $A(s,s')$:

$$\inf_{x\in\mathcal{B}_d(0,\varepsilon)} \inf_{s\in[a,b]} \inf_{s'\in[a',b']} \frac{\mathrm{P}\{A(s,s')\in dx\}}{dx} = \inf_{x\in\mathcal{B}_d(0,\varepsilon)} \inf_{s\in[a,b]} \inf_{s'\in[a',b']} \frac{e^{-|x|^2/2(s+s')}}{[2\pi(s+s')]^{\frac{d}{2}}} = \frac{e^{-\varepsilon^2/2(a+a')}}{[2\pi(b+b')]^{\frac{d}{2}}}.$$

If Ω_d denotes the volume of $\mathcal{B}_d(0,1)$, $\liminf_{\varepsilon\to 0}\varepsilon^{-d}\mathrm{E}\{J_{a,b,a',b'}(\varepsilon)\}$ is at the very least equal to $(b-a)(b'-a')\Omega_d/[2\pi(b+b')]^{\frac{d}{2}}$. Consequently, there exists a strictly positive and finite constant $C_1 = C_1(a,b,a',b',d)$, such that for all $\varepsilon \in (0,1)$,

$$\mathrm{E}\{J_{a,b,a',b'}(\varepsilon)\} \geq C_1\varepsilon^d. \tag{5.1}$$

We only need to remember that C_1 is finite and strictly positive. Next, we estimate the second moment of $J_{a,b,a',b'}(\varepsilon)$, viz.,

$$\begin{aligned}
\mathrm{E}\{J^2_{a,b,a',b'}(\varepsilon)\} &= \int_a^b\int_a^b\int_{a'}^{b'}\int_{a'}^{b'} \mathrm{P}\{|A(s,t)|\leq\varepsilon\ ,\ |A(s',t')|\leq\varepsilon\}\, ds\, ds'\, dt\, dt' \\
&= 2\mathcal{T}_1 + 2\mathcal{T}_2, \qquad \text{where} \\
\mathcal{T}_1 &= \int_a^b\int_s^b\int_{a'}^{b'}\int_t^{b'} \mathrm{P}\{|A(s,t)|\leq\varepsilon\ ,\ |A(s',t')|\leq\varepsilon\}\, ds\, ds'\, dt\, dt', \\
\mathcal{T}_2 &= \int_a^b\int_s^b\int_{a'}^{b'}\int_{a'}^{t} \mathrm{P}\{|A(s,t)|\leq\varepsilon\ ,\ |A(s',t')|\leq\varepsilon\}\, ds\, ds'\, dt\, dt'.
\end{aligned}$$

The estimate for $\mathcal{T}_1$ is straight-forward: when $s \leq s'$ and $t \leq t'$, $A(s',t') = A(s,t) + A'$, where A' is independent of $A(s,t)$ and has the same distribution as $A(s'-s,t'-t)$. This, and the unimodality of the Gaussian distribution, together yield

$$\begin{aligned}
\mathcal{T}_1 &\leq \int_a^b\int_s^b\int_{a'}^{b'}\int_t^{b'} \mathrm{P}\{|A(s,t)|\leq\varepsilon\}\cdot\mathrm{P}\{|A(s'-s,t'-t)|\leq\varepsilon\}\, ds\, ds'\, dt\, dt' \\
&\leq \int_a^b\int_{a'}^{b'} \mathrm{P}\{|A(s,t)|\leq\varepsilon\}\, ds\, dt \cdot \int_0^{b-a}\int_0^{b'-a'} \mathrm{P}\{|A(u,v)|\leq\varepsilon\}\, du\, dv.
\end{aligned}$$

For all $s,t\geq 0$, the probability density of $A(s,t)$ is at most $(s+t)^{-\frac{d}{2}}$. Thus,

$$\mathcal{T}_1 \leq \frac{\Omega_d(b-a)(b'-a')\varepsilon^d}{(b+b')^{\frac{d}{2}}} \cdot \int_0^{b-a}\int_0^{b'-a'} \left[\frac{\Omega_d\varepsilon^d}{(u+v)^{\frac{d}{2}}}\wedge 1\right] du\, dv.$$

A few lines of calculations reveal the existence of a positive and finite constant, $C_2 = C_2(a,b,a',b',d)$, such that for all $\varepsilon \in (0,1)$,

$$\mathcal{T}_1 \le \frac{C_2 \varepsilon^{2d}}{\mathcal{K}_{d-4}(\varepsilon)}. \tag{5.2}$$

The term $\mathcal{T}_2$ satisfies a similar inequality, but showing this requires a small trick. Indeed, when $s \le s'$ and $t \ge t'$, we can write

$$\begin{aligned} A(s,t) &= A(s,t') + Z_1 \\ A(s',t') &= A(s,t') + Z_2, \end{aligned}$$

where $Z_1 = A(s,t) - A(s,t')$ and $Z_2 = A(s',t') - A(s,t')$ are independent from one another, as well as from $A(s,t')$. Note that the probability density function of $A(s,t')$ is bounded above by some finite constant $C_3 = C_3(a,b,a',b',d)$ that can be chosen independently of $s \in [a,b]$ and $t' \in [a',b']$. Therefore,

$$\begin{aligned} \mathcal{T}_2 &\le C_3 \int_a^b \int_s^b \int_{a'}^{b'} \int_{a'}^t \int_{\mathbb{R}^d} \mathrm{P}\{|x+Z_1| \le \varepsilon \ , \ |x+Z_2| \le \varepsilon\}\, dx\, ds\, ds'\, dt\, dt' \\ &= C_3 \int_a^b \int_s^b \int_{a'}^{b'} \int_{a'}^t \int_{\mathcal{B}_d(0,\varepsilon)} \mathrm{P}\{|y+Z_2-Z_1| \le \varepsilon\}\, dy\, ds\, ds'\, dt\, dt' \\ &= C_3 \int_a^b \int_s^b \int_{a'}^{b'} \int_{a'}^t \int_{\mathcal{B}_d(0,\varepsilon)} \mathrm{P}\{|y+A(s',t')-A(s,t)| \le \varepsilon\}\, dy\, ds\, ds'\, dt\, dt' \\ &\le C_3 \Omega_d \varepsilon^d \int_a^b \int_s^b \int_{a'}^{b'} \int_{a'}^t \mathrm{P}\{|A(s',t')-A(s,t)| \le \varepsilon\}\, ds\, ds'\, dt\, dt', \end{aligned}$$

by the unimodality of the Gaussian distribution. A little thought shows that for the s, s', t, t' in question, the distribution of $A(s',t') - A(s,t)$ is the same as that of $A(s'-s, t-t')$. Hence,

$$\mathcal{T}_2 \le C_3 \Omega_d (b-a)(b'-a') \varepsilon^d \int_0^{b-a} \int_0^{b'-a'} \mathrm{P}\{|A(u,v)| \le \varepsilon\}\, du\, dv.$$

From this, and after a few more simple calculations, we conclude the existence of a finite and positive constant $C_4 = C_4(a,b,a',b',d)$, such that for all $\varepsilon \in (0,1)$, $\mathcal{T}_2 \le C_4 \varepsilon^{2d} / \mathcal{K}_{d-4}(\varepsilon)$. Combining this with (5.2), we obtain

$$\mathrm{E}\{J^2_{a,b,a',b'}(\varepsilon)\} \le 2(C_2 + C_4) \frac{\varepsilon^{2d}}{\mathcal{K}_{d-4}(\varepsilon)}. \tag{5.3}$$

Together with Lemma 4.3, equations (5.1) and (5.3) imply that

$$\mathrm{P}\{J_{a,b,a',b'}(\varepsilon) > 0\} \ge \frac{C_1^2}{2(C_2+C_4)} \cdot \mathcal{K}_{d-4}(\varepsilon).$$

On the other hand, if $J_{a,b,a',b'}(\varepsilon) > 0$, then certainly $A([a,b] \times [a',b'])$ intersects $\mathcal{B}_d(0,\varepsilon)$, and the announced probability lower bound of Theorem 5.1 follows.

For the converse inequality, let $\mathcal{F}(s)$ denote the sigma-field generated by $\{B(r);\, r \le s\}$ and $\mathcal{F}'(t)$ the one generated by $\{B'(r);\, r \le t\}$. These give rise to two independent filtrations, based on which we can consider the "two-parameter martingale,"

$$M(s,t) = \mathrm{E}\big[J_{a,2b,a',2b'}(2\varepsilon) \,\big|\, \mathcal{F}(s) \vee \mathcal{F}'(t)\big], \qquad \forall s \in [a,b],\ t \in [a',b']. \tag{5.4}$$

Clearly,

$$\begin{aligned} M(s,t) &\ge \int_s^{2b}\int_t^{2b'} \mathrm{P}\{|A(u,v)| \le 2\varepsilon \,|\, \mathcal{F}(s) \vee \mathcal{F}'(t)\}\, du\, dv \cdot \mathbf{1}_{\mathcal{B}_d(0,\varepsilon)}(A(s,t)) \\ &\ge \int_s^{2b}\int_t^{2b'} \mathrm{P}\{|A(u,v) - A(s,t)| \le \varepsilon \,|\, \mathcal{F}(s) \vee \mathcal{F}'(t)\}\, du\, dv \cdot \mathbf{1}_{\mathcal{B}_d(0,\varepsilon)}(A(s,t)), \end{aligned}$$

thanks to the triangle inequality, and the $\mathcal{F}(s) \vee \mathcal{F}'(t)$-measurability of the random variable $A(s,t)$. On the other hand, whenever $u \ge s$ and $v \ge t$, $A(u,v) - A(s,t)$ has the same distribution as $A(u-s, v-t)$, and is independent of $\mathcal{F}(s) \vee \mathcal{F}'(t)$. (This follows from the Markov property of B and B', together with their mutual independence.) Consequently,

$$\begin{aligned} M(s,t) &\ge \int_s^{2b}\int_t^{2b'} \mathrm{P}\{|A(u-s, v-t)| \le \varepsilon\}\, du\, dv \cdot \mathbf{1}_{\mathcal{B}_d(0,\varepsilon)}(A(s,t)) \\ &\ge \int_0^{b-a}\int_0^{b'-a'} \mathrm{P}\{|A(x,y)| \le \varepsilon\}\, dx\, dy \cdot \mathbf{1}_{\mathcal{B}_d(0,\varepsilon)}(A(s,t)), \end{aligned}$$

for all $s \in [a,b]$ and $t \in [a',b']$. Another simple calculation shows the existence of a nontrivial constant $C_5 = C_5(a,b,a',b',d)$, such that for all $\varepsilon \in (0,1)$, all $s \in [a,b]$, and all $t \in [a',b']$,

$$M(s,t) \ge C_5 \frac{\varepsilon^d}{\mathcal{K}_{d-4}(\varepsilon)} \cdot \mathbf{1}_{\mathcal{B}_d(0,\varepsilon)}(A(s,t)).$$

[Here is why our proof is a sketch: One needs to show that the null set can be chosen to be independent of the choice of $\varepsilon \in (0,1)$, as well as that of $s \in [a,b]$ and $t \in [a',b']$. This can be done, thanks to the regularity theorem of BAKRY (1979).] Thus, we can take suprema over all $s \in [a,b]$ and $t \in [a',b']$ in the previous display, square, and deduce the almost sure inequality:

$$\sup_{s\in[a,b]}\sup_{t\in[a',b']} M^2(s,t) \ge C_5^2 \left[\frac{\varepsilon^d}{\mathcal{K}_{d-4}(\varepsilon)}\right]^2 \cdot \mathbf{1}_{\{A([a,b]\times[a',b'])\cap\mathcal{B}_d(0,\varepsilon)\neq\varnothing\}}.$$

Thus,

$$\mathrm{E}\left[\sup_{s\in[a,b]}\sup_{t\in[a',b']} M^2(s,t)\right] \ge C_5^2 \left[\frac{\varepsilon^d}{\mathcal{K}_{d-4}(\varepsilon)}\right]^2 \cdot \mathrm{P}\{A([a,b]\times[a',b'])\cap\mathcal{B}_d(0,\varepsilon)\neq\varnothing\}. \tag{5.5}$$

Now, $s \mapsto \sup_{t\in[a',b']} M^2(s,t)$ is a submartingale with respect to the filtration $\mathcal{F}$. Thus, by Doob's maximal inequality (REVUZ AND YOR 1991, Theorem 1.7, p. 52),

$$\mathrm{E}\left[\sup_{s\in[a,b]}\sup_{t\in[a',b']} M^2(s,t)\right] \le 4 \sup_{s\in[a,b]} \mathrm{E}\left[\sup_{t\in[a',b']} M^2(s,t)\right].$$

[To be completely rigorous, one needs to look at a suitable augmentation of $\mathcal{F}$ and $\mathcal{F}'$ in order to deduce Doob's inequalities. But such technical details are omitted from our sketch here.] On the other hand, for any fixed $s \in [a,b]$, $t \mapsto M^2(s,t)$ is a submartingale with respect to the filtration $\mathcal{F}'$. Thus, another application of Doob's inequality shows us that

$$\begin{aligned} \mathrm{E}\left[\sup_{s\in[a,b]}\sup_{t\in[a',b']} M^2(s,t)\right] &\leq 16 \sup_{s\in[a,b]}\sup_{t\in[a',b']} \mathrm{E}\left[M^2(s,t)\right] \leq 16\mathrm{E}\left\{J^2_{a,2b,a',2b'}(2\varepsilon)\right\} \\ &\leq 2^{1+2d}(C_2+C_4)\frac{\varepsilon^{2d}}{\mathcal{K}_{d-4}(2\varepsilon)}. \end{aligned}$$

The second inequality follows from Jensen's inequality, and the third from (5.3). This and (5.5) together show that

$$\mathrm{P}\{A([a,b]\times[a',b'])\cap\mathcal{B}_d(0,\varepsilon)\neq\varnothing\} \leq \frac{2^{1+2d}(C_2+C_4)}{C_5^2}\cdot\frac{K^2_{d-4}(2\varepsilon)}{\mathcal{K}_{d-4}(\varepsilon)}.$$

The upper bound of Theorem 5.1 follows easily from this and from the regular variation of the function $\lambda \mapsto \mathcal{K}_\beta(\lambda)$ near $\lambda = 0$, i.e., that there exists C_6 such that for all $\varepsilon \in (0,1)$, $\mathcal{K}_{d-4}(2\varepsilon) \leq C_6\mathcal{K}_{d-4}(\varepsilon)$. □

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Received: 03. 07. 2002
Revised: 29. 07. 2002