ON A FAMILY OF POLYNOMIALS (NOT FOR PUBLICATION)

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1. Introduction

For every integer $n \geq 1$ consider the $2n$-th order polynomial

$$P_n(x) := \sum_{j=0}^{n} \binom{n}{j}^2 x^{2j}(1-x)^{2(n-j)}, \qquad \text{where } x \in (0\,,1). \tag{1.1}$$

Kowalski and Rivoal (2005b) have proved that P_n attains its minimum at $x = 1/2$. That is,

$$P_n(x) \geq P_n(1/2) \qquad \text{for all } x \in (0\,,1). \tag{1.2}$$

Kowalski (2006) has applied this inequality, in turn, to present a novel proof of a slightly weaker form of the celebrated theorem of Paley, Wiener, and Zygmund (1933) in the following form: "Almost every" continuous function $f : [0\,,1] \to \mathbf{R}$ is non-differentiable at almost everywhere point $x \in [0\,,1]$. The first "almost everywhere" holds with respect to Wiener's measure on the space of continuous functions; the second with respect to Lebesgue's measure on $[0\,,1]$.

The derivation of (1.2) by Kowalski and Rivoal (2005b) is involved, as it hinges on a delicate analysis of P_n via its connections to hypergeometric functions ${}_4F_3$ and ${}_3F_2$. Thus, Kowalski (2006) has asked if there are more elementary proofs of (1.2); see also Part (a) of Problem 11155 of the May issue of *The American Mathematical Monthly* (Kowalski and Rivoal, 2005a). The chief aim of this note is to answer this question in the affirmative. This is done by once interpreting the polynomial P_n probabilistically, and once Fourier analytically. As a by-product of our solution we obtain an elementary proof of Part (b) of Problem 11155 of Kowalski and Rivoal (2005a) as well.

Our solution hinges on first establishing the following formula, which can be found already in Kowalski (2006).

Theorem 1.1. *For all $x \in (0\,,1)$,*

$$P_n(x) = \frac{1}{\pi}\int_0^\pi \big(1 - 2x(1-x)(1-\cos t)\big)^n\, dt. \tag{1.3}$$

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Because $x(1-x)$ is maximized at $x = 1/2$, Theorem 1.1 verifies (1.2). Furthermore, this theorem readily implies that $P_n(x+\frac{1}{2}) = \sum_{j=0}^n q_j x^{2j}$, where

$$q_j = \frac{4^j}{\pi}\binom{n}{j}\int_0^\pi \left(\frac{1+\cos t}{2}\right)^{n-j}\left(\frac{1-\cos t}{2}\right)^j dt. \tag{1.4}$$

Thus, if we expand $P_n(x+\frac{1}{2})$ in powers of x, then we find that the coefficient of x^k is nonzero (and positive) if and only if k is an even integer between zero and $2n$. This gives an elementary solution to Problem (b) of Kowalski and Rivoal (2005a). This problem was solved originally in Kowalski and Rivoal (2005b) using more sophisticated ideas.

The forthcoming discussion assumes that the reader is conversant with elements of probability theory at an advanced undergraduate level (Stirzaker, 2003).

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2. **Proof of Theorem 1.1**

For each $x \in (0\,,1)$ let $X_1(x), \ldots, X_n(x)$ denote n independent, identically distributed random variables with common distribution

$$\mathrm{P}\{X_j(x) \neq 0\} = \mathrm{P}\{X_j(x) = 1\} = x \qquad \text{for all } 1 \leq j \leq n. \tag{2.1}$$

Define $S_n(x) := X_1(x) + \cdots + X_n(x)$. Then, a basic fact from elementary probability is that $S_n(x)$ has the *binomial distribution*,

$$\mathrm{P}\{S_n(x) = j\} = \binom{n}{j} x^j (1-x)^{n-j} \qquad \text{for all } j = 0, \ldots, n. \tag{2.2}$$

Define $X_1'(x), \ldots, X_n'(x)$ to be a sequence of independent random variables, all with the same distribution as $X_1(x)$, but also independent of $X_1(x), \ldots, X_n(x)$. Then we can interpret P_n as follows:

$$P_n(x) = \sum_{j=0}^n \mathrm{P}\{S_n(x) = S_n'(x) = j\} = \mathrm{P}\{S_n(x) - S_n'(x) = 0\}, \tag{2.3}$$

where $S_n'(x) := X_1'(x) + \cdots + X_n'(x)$. Define

$$D_j(x) := X_j(x) - X_j'(x). \tag{2.4}$$

A key feature of this interpretation is the fact that

$$S_n(x) - S_n'(x) = \sum_{j=1}^n D_j(x) \tag{2.5}$$

is the sum of n *symmetrized* random variables $D_j(x)$, $j = 1, \dots, n$. "Symmetrization" is meant in the sense of Lévy (1937). Namely, that the distribution of each $D_j(x)$ is the same as that of $-D_j(x)$. Moreover, the said distribution is described as follows:

$$\begin{aligned} &\mathrm{P}\left\{X_j(x) - X_j'(x) = 0\right\} = x^2 + (1-x)^2, \\ &\mathrm{P}\left\{X_j(x) - X_j'(x) = 1\right\} = \mathrm{P}\left\{X_j(x) - X_j'(x) = -1\right\} = x(1-x). \end{aligned} \tag{2.6}$$

If X is a random variable, then its *characteristic function* is the complex-valued function $\psi(t) := \mathrm{E}[e^{itX}]$, as t ranges over all real numbers. A basic fact from Fourier analysis is that if X is integer-valued, then

$$\mathrm{P}\{X = 0\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi(t)\, dt. \tag{2.7}$$

We can verify this by computing the integral directly:

$$\int_{-\pi}^{\pi} \psi(t)\, dt = \int_{-\pi}^{\pi} \mathrm{E}\left[e^{itX}\right]\, dt = \mathrm{E}\left[\int_{-\pi}^{\pi} e^{itX}\, dt\right] = 2\pi \mathrm{E}\left[\mathbf{1}_{\{X=0\}}\right], \tag{2.8}$$

where the random variable $\mathbf{1}_{\{X=0\}}$ takes the values one and zero, depending on whether $X = 0$ or $X = 1$, respectively.[1] Because $\mathrm{E}[\mathbf{1}_{\{X=0\}}] = \mathrm{P}\{X = 0\}$, this proves (2.7).

Equations (2.7), (2.3), and (2.5) together yield the following:

$$\begin{aligned} P_n(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{E}\left[e^{it(S_n(x) - S_n'(x))}\right]\, dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{E}\left[\prod_{j=1}^{n} e^{itD_j(x)}\right]\, dt. \end{aligned} \tag{2.9}$$

The random variables $D_1(x), \dots, D_n(x)$ are independent; and so are their complex exponentials. Because the expectation of the product of independent random variables is the product of the respective expectations, and since $D_1(x), \dots, D_n(x)$ have the same characteristic function, it follows that

$$\begin{aligned} P_n(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\mathrm{E}\left[e^{itD_1(x)}\right]\right)^n\, dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(x^2 + (1-x)^2 + 2x(1-x)\cos t\right)^n\, dt. \end{aligned} \tag{2.10}$$

A direct computation concludes the proof. □

[1] The interchange of expectation and the integral is justified by Fubini's theorem, because $|\psi(t)| \le 1$ for all values of t.

3. **A remark**

As Kowalski (2006) points out,

$$P_n(1/2) \sim \frac{1}{\sqrt{\pi n}} \quad \text{as } n \to \infty, \tag{3.1}$$

where "$a_n \sim b_n$" means that a_n/b_n converges to one as $n \to \infty$. [This is an example of the so-called "local limit theorem" of classical probability theory.] He also derives the quantitative bound

$$P_n(1/2) \geq \frac{9}{32\sqrt{n}} \approx \frac{0.28125}{\sqrt{n}} \qquad \text{for all } n \geq 1. \tag{3.2}$$

See Lemma 5.1 of that paper (*loc. cit.*). Next we observe that the Fourier-analytic method here improves (3.2), without much effort, to the following:

$$P_n(1/2) \geq \frac{4}{3\pi\sqrt{n}} \approx \frac{0.42441}{\sqrt{n}} \qquad \text{for all } n \geq 1. \tag{3.3}$$

This is nearly unimprovable because we will see soon in (3.4) that $P_1(1/2) = 1/2$. But the lower bound can be improved for larger values of n. For instance, the same method below will imply that $P_n(1/2) \geq 43/(30\pi\sqrt{n}) \approx 0.45624/\sqrt{n}$ for all $n \geq 2$.

Now let us prove (3.3). In accord with (2.10), and by symmetry,

$$P_n(1/2) = \frac{1}{\pi}\int_0^\pi \left(\frac{1+\cos t}{2}\right)^n dt. \tag{3.4}$$

Taylor's expansion of the cosine, with remainder, implies that $\frac{1}{2}(1+\cos t)$ is greater than or equal to the maximum of $1-(t^2/4)$ and zero. Hence,

$$\begin{aligned} P_n(1/2) &\geq \frac{1}{\pi}\int_0^2 \left(1-\frac{t^2}{4}\right)^n dt \\ &\geq \frac{1}{\pi}\int_0^{2/\sqrt{n}} \left(1-\frac{t^2}{4}\right)^n dt \\ &= \frac{1}{\pi\sqrt{n}}\int_0^2 \left(1-\frac{s^2}{4n}\right)^n ds. \end{aligned} \tag{3.5}$$

A basic fact, from calculus, is that $(1-\theta/n)^n$ increases with n for all θ between zero and one. This and the preceding together prove that

$$P_n(1/2) \geq \frac{1}{\pi\sqrt{n}}\int_0^2 \left(1-\frac{s^2}{4}\right) ds, \tag{3.6}$$

which yields (3.3) readily.

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