

RESEARCH STATEMENT

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1. BACKGROUND AND HISTORY

My area of interest is Algebraic Geometry. I am mainly interested in the Birational Geometry of algebraic varieties, namely, the Minimal Model Program and the singularities of the MMP in characteristic 0 and $p > 0$.

One of the main goals of algebraic geometry is to classify algebraic objects, namely algebraic varieties. In birational geometry we want to classify algebraic varieties up to birational isomorphisms, i.e., we say that two algebraic varieties are “Birationally Equivalent” if their function fields are isomorphic. This defines an equivalence relation on the class of all algebraic varieties of fixed dimension. Next we want to find a “Good” representative or Model, for each of these equivalence classes.

In dimension 1, for any curve C we know that its normalization \tilde{C} is smooth and it is birational to C , so we can pick \tilde{C} as the “Good” representative for the equivalence class of C . Here \tilde{C} is smooth and this is as good as it gets for a “Good” representative.

In dimension 2, for a surface S , its normalization is not smooth in general, so choosing a good representative in this case is not as easy as in the curve case. In the 20th century the Italian School of Algebraic Geometry showed that it is possible to choose a “Good” model for surfaces. At that time Castelnuovo showed that given a smooth projective algebraic surface X , if C is a (-1) -curve on X then C can be contracted to get another smooth projective surface Y which is birational to X . We can continue this process of contracting (-1) -curves. It can be shown that this process stops after a finitely many steps producing a smooth projective surface X^* which does not contain any (-1) -curves. Now if the canonical divisor K_X of the surface X is pseudoeffective, then K_{X^*} becomes *nef*. This X^* enjoys a special property: Every birational morphism $f : X^* \rightarrow Z$ to another smooth projective surface Z is necessarily an isomorphism. We call X^* a Minimal Model and it is our candidate for the “Good” representative in the surface case.

In higher dimension ($\dim \geq 3$) one would like to use similar methods to find minimal models. It is not hard to see that in higher dimension we can not work in the category of smooth varieties alone, we must allow some mild singularities. Constructing minimal models in higher dimensions eluded mathematicians for more than half of the 20th century until 1988 when Mori gave an explicit construction of flips, the last ingredient necessary to construct the minimal model for threefolds. The corresponding algorithm

called the “Minimal Model Program (MMP)” or “Mori Program”, which aims to obtain a minimal model for varieties in arbitrary dimension via a sequence of divisorial contractions and flips. By blowing-down K_X -negative curves, we may get a divisorial contraction or a flipping contraction. To deal with the case of a flipping contraction $f : X \rightarrow Z$, we need to show that flip exists, i.e., that the canonical algebra

$$\bigoplus_{m \geq 0} f_* \mathcal{O}_X(m(K_X + \Delta))$$

is a finitely generated \mathcal{O}_Z -algebra. Two obstacles arise in this program:

- (1) Existence of flips,
- (2) Termination of a sequence of flips.

In [Mor88] Mori showed the existence of flips for 3-dimensional varieties over \mathbb{C} , answering the first question and hence finishing the MMP in dimension 3 (the second problem is not too difficult in this case). Existence of flips for $\dim \geq 4$ remained open until 2006 when Birkar, Cascini, Hacon, McKernan proved the existence of flips in arbitrary dimension over \mathbb{C} in their celebrated paper [BCHM10], hence answering the First Question once and for all over \mathbb{C} . In the same article they also proved the termination of flips with scaling for varieties of log general type and hence proving the existence of the minimal model for those varieties in arbitrary dimensions over \mathbb{C} . This answers the second question for a large class of varieties over \mathbb{C} , but the general question of termination of flips remain open till this date.

The next obvious question one can ask is: Does MMP work in positive characteristic? Very little answers are known for this question. In dimension 1 and 2 and for smooth varieties, the same methods as characteristic 0 work in positive characteristic. For the MMP on singular surfaces in $\text{char } p > 0$ see [KK] and [FT12].

There are two main technical difficulties that appear in the positive characteristic MMP:

- (1) Resolution of singularities in higher dimension (≥ 4) is not known in characteristic $p > 0$.
- (2) The Kawamata-Viehweg vanishing theorem fails in higher dimension (≥ 2) in characteristic $p > 0$.

Fortunately the resolution of singularities is known in dimension 3 and in $\text{char } p > 0$ due to [Abh98], [Cut04], [CP08, CP09]. Kawamata-Viehweg vanishing theorem is a very essential tool for char 0 MMP, almost any proof related to MMP uses this vanishing theorem to lift sections of a line bundle from a divisor. For example, lifting sections is in the heart of the proof of the main theorem in [BCHM10]. Their technique is based on ideas of Siu and repeated and systematic use Kawamata-Viehweg vanishing theorem. Recently in a series of papers by Hara, Hochster, Huneke, Mustařá, Schwede, Smith, Tucker and others, it has become clear that one can sometimes replace the vanishing theorems by use of test ideals, Frobenius maps and the Serre vanishing theorem. The F -singularity techniques coming from the tight closure theory in commutative algebra

have proved to be a powerful tool in studying birational geometry in char $p > 0$.

Good progress has been made recently towards the minimal model program on 3-folds in char $p > 5$, due to Hacon, Xu, Birkar and Waldron, Schwede, Gongyo, Tanaka, Cascini and others (see [HX13], [Bir13], [BW14] and [CGS14]). They successfully managed to use the existence of the resolution of singularities for 3-folds and the F -singularity techniques to run the MMP on 3-folds in char $p > 5$. In dimension ≥ 4 almost nothing is known towards the minimal model program in char $p > 0$.

2. PAST RESEARCH

My past research involved studies of singularities of the MMP in char $p > 0$. In [Das13] I showed that a char $p > 0$ analog of the log terminal inversion of adjunction holds in arbitrary dimension. More specifically I proved the following theorem:

Theorem 2.1. [Das13, Theorem 4.1, Corollary 5.4] *Let $(X, S + B)$ be a pair where X is a normal variety in characteristic $p > 0$, $S + B \geq 0$ is a \mathbb{Q} -divisor, $K_X + S + B$ is \mathbb{Q} -Cartier and $S = \lfloor S + B \rfloor$. Let $\nu : S^n \rightarrow S$ be the normalization and write $(K_X + S + B)|_{S^n} = K_{S^n} + B_{S^n}$. If (S^n, B_{S^n}) is strongly F -regular then S is normal, furthermore S is a unique center of sharp F -purity of $(X, S + B)$ in a neighborhood of S and $(X, S + B)$ is purely F -regular near S .*

Inversion of adjunction is a very powerful and essential tool to run the MMP. It allows us to get specific information about the singularities of the ambient varieties from the singularities of its subvarieties. The above mentioned inversion of adjunction is first proved in [HX13, 4.1] using resolution of singularities, so in particular their proof works up to dimension 3. In [Das13] I proved the same result but without using the resolution of singularities, hence proving the result in arbitrary dimension. This inversion of adjunction is used in [HX13] and [Bir13] to construct flips for 3-folds in char $p > 5$. Also, since the minimal model program implies inversion of adjunction, it is an important test case for the MMP.

In [Das13] I also proved the equality of the ‘Different’ and the ‘ F -Different’ conjectured by Schwede in [Sch09]. The ‘Different’ was originally defined by Shokurov as a correction term for the adjunction formula in singular varieties and the F -Different is defined by Schwede as a correction term for the F -adjunction coming from the F -singularities. I proved the following theorem:

Theorem 2.2. [Das13, Theorem 5.3] *Let $(X, S + \Delta \geq 0)$ be a pair, where X is a normal excellent scheme of pure dimension over a field k of characteristic $p > 0$ and $S + \Delta \geq 0$ is a \mathbb{Q} -divisor on X such that $(p^e - 1)(K_X + S + \Delta)$ is Cartier for some $e > 0$. Also assume that S is a reduced Weil divisor and $S \wedge \Delta = 0$. Then the F -Different, $F\text{-Diff}_{S^n}(\Delta)$ is equal to the Different, $\text{Diff}_{S^n}(\Delta)$, i.e., $F\text{-Diff}_{S^n}(\Delta) = \text{Diff}_{S^n}(\Delta)$, where $S^n \rightarrow S$ is the normalization morphism.*

The equality of these two technical terms creates a bridge between the MMP singularities and the F -singularities in char $p > 0$. It allows one to apply the results obtained in F -singularities to the MMP singularities. and vice-versa. In [CGS14], equality of these two terms has been used to prove that a certain KLT pair is relatively globally F -regular.

In a current paper [DH] jointly with Christopher Hacon we proved the adjunction formula for codimension 2 minimal LC centers of a \mathbb{Q} -factorial 3-fold in char $p > 5$. In the same article we also prove a relative vanishing theorem and the normality of minimal log canonical centers. More specifically we prove the following theorems:

Theorem 2.3. *Let (X, Δ) be a \mathbb{Q} -factorial 3-fold log canonical pair with isolated center W , and S , a unique exceptional divisor of discrepancy -1 over W . Let $f : (Y, S) \rightarrow (X, W)$ be the corresponding divisorial extraction such that $K_Y + S + B = f^*(K_X + \Delta)$. Then $R^1 f_* \mathcal{O}_Y(-S) = 0$.*

We use this theorem to replace the Kawamata-Viehweg vanishing theorem in our article. In particular we use it to prove the normality of the minimal log canonical center, which is the following theorem:

Theorem 2.4. *Let (X, Δ) be a \mathbb{Q} -factorial 3-fold log canonical pair such that all log canonical centers are contained in Δ . If W is a minimal log canonical center of (X, Δ) then it is normal.*

Finally we prove the adjunction formula on codimension 2 minimal log canonical centers in char $p > 5$.

Theorem 2.5. *Let $(X, D > 0)$ be a \mathbb{Q} -factorial 3-fold log canonical pair such that all of the log canonical centers are contained in D . Let W be a minimal log canonical center of (X, D) , and codimension of W is 2. Then the following hold:*

- (1) *There exist canonically determined effective \mathbb{Q} -divisors M_W and D_W on W such that $(K_X + D)|_W \sim_{\mathbb{Q}} K_W + D_W + M_W$. Moreover, if $D = D' + D''$ with D' (resp. D'') the sum of all irreducible components which contain (resp. do not contain) W , then M_W is determined only by the pair (X, D') .*
- (2) *There exists an effective \mathbb{Q} -divisor M'_W such that $M'_W \sim_{\mathbb{Q}} M_W$ and the pair $(W, D_W + M'_W)$ is KLT.*

3. FUTURE PROJECTS

To get a better understanding of the MMP in char $p > 0$, we must have a good understanding of the singularities of MMP. In char 0 we have quite a good understanding of these singularities, however in char $p > 0$ the picture is not so clear. From the work of Hara, Watanabe, Takagi, Schwede, Smith, Tucker, Patakfalvi, Blickle, Yoshida, Fedder, Mustață, Metha, Srinivas, Zhang and others, it has become clear that F -singularities are closely related to the MMP singularities. In [Das13] I proved a char $p > 0$ analog of the log terminal inversion of adjunction. In relation to this result

next I want to study the char $p > 0$ analog of the log canonical inversion of adjunction. More specifically I would like to prove the following statement:

Project 3.1. *Let $(X, \Delta \geq 0)$ be pair such that X is a normal variety in characteristic $p > 0$ and $K_X + \Delta$ is \mathbb{Q} -Cartier. Let W be a sharp F -pure center of $(X, S + B)$. Assume W is normal and $(K + \Delta)|_W \sim_{\mathbb{Q}} K_W + \Delta_W$ is defined by F -adjunction. If (W, Δ_W) is sharply F -pure then (X, Δ) is sharply F -pure near W .*

Partial answers to this problem are known. In [Sch09], Schwede proved that the result holds when the index of $K_X + \Delta$ is not divisible by the characteristic p . So the only unknown case is when the index of $K_X + \Delta$ is divisible by p . In this case Frobenius techniques seem to not work as it naturally imposes the non divisibility condition on the index of $K_X + \Delta$. I plan to use geometric perturbation techniques as in [Das13] to prove this result.

I plan to work on another project quite similar to the previous one. I would like to prove the log canonical inversion of adjunction in 3-folds in char $p > 5$. In char 0 this result is due to Kawakita [Kaw07] for centers of codimension 1 and Eisenstein [Eis11] and Hacon [Hac14] for higher codimensions. In 3-folds and in char $p > 5$, the codimension 1 case is known, due to [HX13, 6.2] and [Bir13]. So the main question is the higher codimensional case. More specifically I plan to prove the following result:

Project 3.2. *Let W be a log canonical center of a pair $(X, \Delta = \sum \delta_i \Delta_i)$, where $0 \leq \delta_i \leq 1$ and X is a 3-fold in characteristic $p > 5$. Then (X, Δ) is log canonical on a neighborhood of W if and only $(W^n, \mathbf{B}(W; X, \Delta))$ is log canonical.*

In light of very recent developments on the MMP for 3-folds in char $p > 5$ ([HX13], [Bir13] and [BW14]), we know that MMP with scaling works for 3-folds in char $p > 5$. Keeping this in mind I would like to use the similar techniques as in [Hac14] to prove this result.

In another project I would like to generalize the Theorem 2.5 mentioned above by removing the hypothesis that X is \mathbb{Q} -factorial and KLT. More specifically I would like to prove the following statement:

Project 3.3. *Let (X, Δ) be a 3-fold log canonical pair in char $p > 5$ and W , a codimension 2 log canonical center of (X, Δ) . Then the adjunction formula holds for W^n , where $W^n \rightarrow W$ is the normalization.*

One of the main challenges in this problem is that the Kodaira's canonical bundle formula for elliptic fibration is not well behaved in char $p > 0$, due to the existence of fibers with wild ramification. I plan to overcome this difficulties by explicitly classifying these wild ramifications.

In a series of recent papers [HX13], [Bir13] and [BW14] by Hacon, Xu, Birkar and Waldron, the existence of flips and the existence of minimal model is been established

for any KLT 3-fold in char $p > 5$. The restriction on the characteristic of X comes from a theorem of Hara, which says, a normal surface S in char $p > 5$ is KLT if and only if it is strongly F -regular. This result is known to fail in char < 5 .

I would like to fill the gap in the MMP in char $p < 5$. More specifically I would like to prove the following result:

Project 3.4. *Flips exist on 3-folds in char $p < 5$.*

It seems that the general theory of test ideal based vanishing theorem is not sufficient in char $p < 5$. I plan to classify all the cases where Hara's theorem fail. I believe that explicit classifications is possible for each case of char $p < 5$ and hope to use this classification to construct flips in the missing dimensions.

Finally I would like to prove termination of flips for 3-folds in char $p > 0$.

Project 3.5. *Any sequence of flips on 3-folds in characteristic $p > 0$ terminates.*

Partial answers to this question are known, for example, if (X, Δ) has terminal singularities then the termination of flips is straight forward. So the non-trivial case is the termination of KLT flips. I hope to use some recent results on the study of minimal log discrepancies and an approach due to Shokurov.

REFERENCES

- [Abh98] S. S. Abhyankar. *Resolution of singularities of embedded algebraic surfaces*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, second edition, 1998.
- [BCHM10] Caucher Birkar, Paolo Cascini, Christopher D. Hacon, and James McKernan. Existence of minimal models for varieties of log general type. *J. Amer. Math. Soc.*, 23(2):405–468, 2010.
- [Bir13] Caucher Birkar. Existence of flips and minimal models for 3-folds in char p . *ArXiv e-prints*, November 2013.
- [BW14] C. Birkar and J. Waldron. Existence of Mori fibre spaces for 3-folds in char $p > 5$. *ArXiv e-prints*, October 2014.
- [CGS14] P. Cascini, Y. Gongyo, and K. Schwede. Uniform bounds for strongly F -regular surfaces. *ArXiv e-prints*, January 2014.
- [CP08] Vincent Cossart and Olivier Piltant. Resolution of singularities of threefolds in positive characteristic. I. Reduction to local uniformization on Artin-Schreier and purely inseparable coverings. *J. Algebra*, 320(3):1051–1082, 2008.
- [CP09] Vincent Cossart and Olivier Piltant. Resolution of singularities of threefolds in positive characteristic. II. *J. Algebra*, 321(7):1836–1976, 2009.
- [Cut04] Steven Dale Cutkosky. *Resolution of singularities*, volume 63 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2004.
- [Das13] Omprokash Das. On Strongly F -Regular Inversion of Adjunction. *ArXiv e-prints*, October 2013.
- [DH] Omprokash Das and Christopher D. Hacon. On the Adjunction Formula on 3-folds in Characteristic p . *In preparation*.
- [Eis11] E. Eisenstein. Inversion of subadjunction and multiplier ideals. *ArXiv e-prints*, April 2011.
- [FT12] Osamu Fujino and Hiromu Tanaka. On log surfaces. *Proc. Japan Acad. Ser. A Math. Sci.*, 88(8):109–114, 2012.

- [Hac14] Christopher D. Hacon. On the log canonical inversion of adjunction. *Proc. Edinb. Math. Soc. (2)*, 57(1):139–143, 2014.
- [HX13] Christopher D. Hacon and Chenyang Xu. On the three dimensional minimal model program in positive characteristic. *ArXiv e-prints*, February 2013.
- [Kaw07] Masayuki Kawakita. Inversion of adjunction on log canonicity. *Invent. Math.*, 167(1):129–133, 2007.
- [KK] János Kollár and Sándor Kovács. Birational geometry of log surfaces. *preprint*.
- [Mor88] Shigefumi Mori. Flip theorem and the existence of minimal models for 3-folds. *J. Amer. Math. Soc.*, 1(1):117–253, 1988.
- [Sch09] Karl Schwede. F -adjunction. *Algebra Number Theory*, 3(8):907–950, 2009.