ON THE ADJUNCTION FORMULA ON 3-FOLDS IN CHARACTERISTIC $p$

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Abstract. In this article we prove a relative Kawamata-Viehweg vanishing-type theorem for plt 3-folds in characteristic $p > 5$. We use this to prove the normality of minimal log canonical centers and the adjunction formula for codimension 2 subvarieties on $\mathbb{Q}$-factorial 3-folds in characteristic $p > 5$.

1. Introduction

Let $(X, \Delta)$ be a log canonical pair and $W$ a minimal log canonical center, then (under mild technical assumptions) by Kawamata’s celebrated subadjunction theorem, it is known that $W$ is normal and we can write $(K_X + \Delta)|_W = K_W + \Delta_W$ where $(W, \Delta_W)$ is Kawamata log terminal [Kaw98] (see also [Kaw97], [Kaw97] and the references therein). The proof of this result is based on the Kawamata-Viehweg vanishing theorem and Hodge theory. These results are known to fail in characteristic $p > 0$ and therefore one may expect that Kawamata’s subadjunction also fails in this context. It should however be noted that related results have been obtained in the closely related context of $F$-singularities (see for example [Sch09], [HX13] and [Das13]) and that the minimal model program has been established for 3-folds in characteristic $p > 5$ (see [HX13] and [Bir13]). In particular [HX13] exploits the fact that plt singularities in dimension 3 and characteristic $p > 5$ are closely related to the analogous notion of $F$-plt singularities. In this paper, using the results from [HX13] and [Bir13], we show that in dimension 3 and characteristic $p > 5$ a relative version of the Kawamata-Viehweg vanishing theorem holds and we use this to establish that (under some mild technical conditions) the analog of Kawamata’s subadjunction result holds.

Theorem (Theorem 2.5). Let $(X, \Delta \geq 0)$ be a $\mathbb{Q}$-factorial 3-fold log canonical pair with isolated center $W$, $\text{codim}_X W = 2$ and $S$, a unique exceptional divisor $E$ dominating $W$ with $a(E, X, \Delta) = -1$. Also assume that $X$ has KLT singularities. Let $f : (Y, S + B) \rightarrow (X, W)$ be the corresponding divisorial extraction such that $K_Y + S + B = f^*(K_X + \Delta)$. Then $R^1 f_* \mathcal{O}_Y(-S) = 0$.

This result allows us to prove the normality of the minimal LC centers for 3-folds.
**Theorem** (Theorem 2.6). Let \((X, \Delta)\) be a \(\mathbb{Q}\)-factorial 3-fold log canonical pair such that \(X\) has Kawamata log terminal singularities. If \(W\) is a minimal log canonical center of \((X, \Delta)\), then \(W\) is normal. If moreover the coefficients of \(D\) belong to a DCC set \(I \subseteq [0, 1]\) and \(\text{Char}(k) > 2 \delta\), where \(\delta > 0\) is the minimum of the set \(D(I) \cap (0, 1]\) (where \(D(I)\) is defined in 3.1). Then the following hold:

1. There exists effective \(\mathbb{Q}\)-divisors \(\Delta_W\) and \(M_W\) on \(W\) such that \((K_X + \Delta)|_W \sim \mathbb{Q} K_W + \Delta_W + M_W\). Moreover, if \(\Delta = \Delta' + \Delta''\) with \(\Delta'\) (resp. \(\Delta''\)) the sum of all irreducible components which contain (resp. do not contain) \(W\), then \(M_W\) is determined only by the pair \((X, \Delta')\).
2. There exists an effective \(\mathbb{Q}\)-divisor \(M'_W\) such that \(M'_W \sim \mathbb{Q} M_W\) and the pair \((W, \Delta_W + M'_W)\) is KLT.

All of the results in this article hold in characteristic \(p > 5\) unless stated otherwise. We will use the standard terminologies and notations from [KM98]. We also use the abbreviations: LC for log canonical, KLT for Kawamata log terminal, PLT for purely log terminal, DLT for divisorially log terminal, NLC for non-log canonical centers, and NKLT centers for non-Kawamata log terminal centers. If \((X, \Delta)\) is LC, then the NKLT centers are also known as log canonical centers or LC centers.

## 2. Relative Vanishing Theorem and Minimal Log Canonical Centers

In this section we will prove a relative vanishing theorem and then use it to prove the normality of minimal log canonical centers.

**Lemma 2.1.** Let \(X\) be a \(\mathbb{Q}\)-factorial KLT 3-fold and \((X, \Delta \geq 0)\), a log canonical pair. Let \(W_1\) and \(W_2\) be two log canonical centers of \((X, \Delta)\). Then every irreducible component of \(W_1 \cap W_2\) is a log canonical center of \((X, \Delta)\).

**Proof.** There are three cases depending on the codimension of \(W_1\) and \(W_2\).

**Case I:** \(\text{codim}_X W_1 = \text{codim}_X W_2 = 1\). In this case \(W_1\) and \(W_2\) are components of \(\Delta\). Let \(\Delta = W_1 + W_2 + \Delta\). Then by adjunction we have

\[(K_X + W_1 + W_2 + \Delta)|_{W_1^n} = K_{W_1^n} + \text{Diff}_{W_1}(\Delta) + W_2|_{W_1^n},\]

where \(W_1^n \to W_1\) is the normalization. By localizing at the generic point of an irreducible component of \(W_1 \cap W_2\) we reduce it to a surface problem. Now, on a surface in characteristic \(p > 0\), the relative Kawamata-Viehweg vanishing and Kollar’s connectedness theorem hold (see [Kol13, 10.13] and [Das13, 3.1]). Thus on a surface the intersection of two LC centers is a LC center and we are
done.

**Case II**: codim\(_X W_1 = 1\) and codim\(_X W_2 = 2\). Since \(X\) is \(\mathbb{Q}\)-factorial, \((X, (1-\epsilon)\Delta)\) is KLT for any \(0 < \epsilon < 1\). Thus by [Bir13] there exists a \(\mathbb{Q}\)-factorial model \(f' : X' \to (X, \Delta)\) of relative Picard number \(\rho(X'/X) = 1\) such that \(\text{Ex}(f')\) is a unique exceptional divisor \(E'\) over \(W_2\) and

\[
(2.1) \quad K_{X'} + E' + W'_1 + \Delta' = f'^*(K_X + \Delta)
\]

where \(\Delta' \geq 0\), and \(W'_1\) is the strict transform of \(W_1\) under \(f'\).

Since \(W'_1\) and \(E'\) are \(\mathbb{Q}\)-Cartier, they intersect along a curve (possibly reducible). Let \(C'\) be an irreducible component of \(W'_1 \cap E'\). Then by Case I, \(C'\) is a LC center of \((X', E' + W'_1 + \Delta' \geq 0)\). Since every irreducible component of \(W'_1 \cap W_2\) is dominated by an irreducible component of \(W'_1 \cap E'\), we are done by relation \((2.1)\).

**Case III**: codim\(_X W_1 = \text{codim}_X W_2 = 2\). Again since \(X\) is \(\mathbb{Q}\)-factorial, \((X, (1-\epsilon)D)\) is KLT for any \(0 < \epsilon < 1\). Thus by [Bir13] there exists a \(\mathbb{Q}\)-factorial model \(f' : X' \to (X, \Delta)\) such that \(\text{Ex}(f') = E'_1 \cup E'_2\), where \(f'(E'_1) = W_1\) and \(f'(E'_2) = W_2\), and

\[
(2.2) \quad K_{X'} + E'_1 + E'_2 + \Delta' = f'^*(K_X + \Delta).
\]

Since \(E'_1\) and \(E'_2\) are \(\mathbb{Q}\)-Cartier, they intersect along a curve (possibly reducible). Let \(C'\) be an irreducible component of \(E'_1 \cap E'_2\). Then by Case I, \(C'\) is a LC center of \((X', E'_1 + E'_2 + \Delta' \geq 0)\). Since every irreducible component of \(W'_1 \cap W_2\) is dominated by an irreducible component of \(E'_1 \cap E'_2\), we are done by relation \((2.2)\).

The following proposition is a characteristic \(p > 5\) version of Fujino’s adjunction theorem for DLT pairs (see [Cor07, 3.9.2] and [Kol13, 4.16]) on a \(\mathbb{Q}\)-factorial 3-fold.  

**Proposition 2.2** (DLT Adjunction). Let \((X, \Delta \geq 0)\) be a \(\mathbb{Q}\)-factorial DLT \(n\)-fold with \(n \leq 3\) such that \(\Delta = D_1 + D_2 + \cdots + D_r + B\) and \(|\Delta| = D_1 + D_2 + \cdots + D_r\). Assume that \(X\) has KLT singularities. Then following hold:

1. The \(s\)-codimensional log canonical centers of \((X, \Delta)\) are exactly the irreducible components of the various intersections \(D_{i_1} \cap \cdots \cap D_{i_s}\) for some \(\{i_1, \ldots, i_s\} \subset \{1, \ldots, r\}\).
2. Every irreducible component of \(D_{i_1} \cap \cdots \cap D_{i_s}\) is normal and of pure codimension \(s\).
(3) Let \( W \) be a log canonical center of \((X, \Delta)\), then there exists an effective \( \mathbb{Q} \)-divisor \( \Delta_W \geq 0 \) on \( W \) such that \((K_X + \Delta)|_W \sim_{\mathbb{Q}} K_W + \Delta_W \) and \((W, \Delta_W)\) is DLT.

(4) If \( D_i \cap D_j = \emptyset \) for all \( i \neq j \), then \((X, \Delta)\) is in fact PLT.

Proof. The result is well known in dimension \( \leq 2 \). (1) follows from the proof in [Kol13, Theorem 4.16].

Since \( X \) is \( \mathbb{Q} \)-factorial, \((X, D_i)\) is also PLT and then by adjunction \((D_i, \text{Diff}_{D_i})\) is KLT. Since \( \text{Diff}_{D_i} \) has standard coefficients, by [Har98] and [HX13, 3.1], \((D_i, \text{Diff}_{D_i})\) is strongly F-regular in characteristic \( p > 5 \). Then by [HX13, 4.1] and [Das13, 4.1, 5.4], \( D_i \) is normal. This proves that every irreducible component of \( \lfloor \Delta \rfloor \) is normal and hence (2) holds for \( s = 1 \).

It is easy to see that \( (D, \text{Diff}_D(\Delta - D)) \) is DLT, and so \( D_i \) is a \( \mathbb{Q} \)-factorial surface by [FT12, 6.3]. (2) and (3) now follow from the result in dimension 2. (4) is immediate. \( \square \)

Definition 2.3 (Divisorial Extraction). Let \((X, \Delta \geq 0)\) be a \( \mathbb{Q} \)-factorial 3-fold log canonical pair with a unique exceptional divisor \( E \) over \( X \) of discrepancy \( a(E, X, \Delta) = -1 \). A divisorial extraction is a \( \mathbb{Q} \)-factorial PLT model \( f : (Y, E + \Delta') \rightarrow (X, \Delta) \) of relative Picard number \( \rho(Y/X) = 1 \), such that \( K_Y + E + \Delta' = f^*(K_X + \Delta) \).

Remark 2.4. Divisorial extractions exist in any dimension in characteristic 0 by [BCHM10] and [KK10, 3.1], and in dimension 3 and in characteristic \( p > 5 \) by [HX13] and [Bir13].

Theorem 2.5 (Relative Vanishing Theorem). Let \((X, \Delta \geq 0)\) be a \( \mathbb{Q} \)-factorial 3-fold log canonical pair with isolated center \( W \), \( \text{codim}_X W = 2 \) and \( S \), a unique exceptional divisor dominating \( W \) with \( a(S, X, \Delta) = -1 \). Also assume that \( X \) has KLT singularities. Let \( f : (Y, S + B) \rightarrow (X, W) \) be the corresponding divisorial extraction such that \( K_Y + S + B = f^*(K_X + \Delta) \). Then \( R^1f_*\mathcal{O}_Y(-S) = 0 \).

Proof. Note that \( -S \) is \( \mathbb{Q} \)-Cartier \( f \)-ample divisor and since \( X \) is klt, \( W \) is contained in the support of \( \delta \) and hence \( B \cap S \neq \emptyset \).

Claim: The following sequence is exact at all codimension 2 points of \( Y \nolabel{add ref}

\begin{align*}
0 \rightarrow \mathcal{B}_e \longrightarrow F^e_*\mathcal{O}_Y((1 - p^e)K_Y - p^e S) \longrightarrow \mathcal{O}_Y(-S) \longrightarrow 0
\end{align*}

for all \( e \gg 0 \) and sufficiently divisible, where \( \phi_e \) is defined as in \( ^1 \) and \( \mathcal{B}_e \) is the kernel of \( \phi_e \).
Granting Claim (2.3) for the time being, we will show that \( R^1 f_* \mathcal{O}_Y(-S) = 0. \)

The exact sequence (2.3) can be split into the following two exact sequences

\[
\begin{align*}
(2.4) & \quad 0 \to \mathcal{B}_e \to F_*^e \mathcal{O}_Y((1 - p^e)K_Y - p^e S) \xrightarrow{\phi_e} \text{Im}(\phi_e) \to 0 \\
(2.5) & \quad 0 \to \text{Im}(\phi_e) \to \mathcal{O}_Y(-S) \to \mathcal{Q}_e \to 0
\end{align*}
\]

where \( \mathcal{Q}_e \) is the corresponding quotient.

Pushing forward the exact sequence (2.4) by \( f \) we get

\[
(2.6) \quad R^1 f_* (F_*^e \mathcal{O}_Y((1 - p^e)K_Y - p^e S)) \to R^1 f_* \text{Im}(\phi_e) \to R^2 f_* \mathcal{B}_e.
\]

Now \( R^2 f_* \mathcal{B}_e = 0 \), since the maximum dimension of the fibers of \( f \) is 1.

Let \( r \) be the index of \( K_Y + S \) and \( H = -(K_Y + S) \). By the division algorithm there exist integers \( k > 0 \) and \( 0 \leq b < r \) such that \( (p^e - 1) = r \cdot k + b \). Then by Serre vanishing

\[
R^1 f_* (F_*^e \mathcal{O}_Y((1 - p^e)K_Y - p^e S)) = F_*^e (R^1 f_* \mathcal{O}_Y(k \cdot r H - b(K_Y + S) - S)) = 0
\]

for all \( e \gg 0 \) and sufficiently divisible, since \( H \) is \( f \)-ample (since \( B \cap S \neq \emptyset \)).

Thus from (2.6) we get

\[
(2.7) \quad R^1 f_* \text{Im}(\phi_e) = 0.
\]

Again, pushing forward the exact sequence (2.5) we get

\[
(2.8) \quad R^1 f_* \text{Im}(\phi_e) \to R^1 f_* \mathcal{O}_Y(-S) \to R^1 f_* \mathcal{Q}_e.
\]

\( R^1 f_* \mathcal{Q}_e = 0 \), since \( \mathcal{Q}_e \) is supported at finitely many points and \( R^1 f_* \text{Im}(\phi_e) = 0 \) by (2.7). Thus we have

\[
(2.9) \quad R^1 f_* \mathcal{O}_Y(-S) = 0.
\]

We will now prove Claim (2.3). By Proposition 2.2, \( S \) is normal and \((Y, S + B)\) is PLT. Since \( Y \) is \( \mathbb{Q} \)-factorial, \((Y, S)\) is also PLT.

Now, since the question is local on \( Y \), we may assume that \( Y \) is affine. Then by [HX13, 2.13] we can choose an effective \( \mathbb{Q} \)-Cartier divisor \( G \geq 0 \) not containing \( S \) and with sufficiently small coefficients such that \( K_Y + S + G \) is \( \mathbb{Q} \)-Cartier with index not divisible by \( p \).

Localizing \( Y \) at a codimension 2 point we may assume that \( Y \) is an excellent surface. Thus by adjunction we have \((K_Y + S + G)|_S = K_S + D_S + G|_S\), where \( D_S \) is the Different. Since \((Y, S)\) is PLT, \((S, D_S)\) is KLT by adjunction. Hence
$(S, D_S)$ is strongly $F$-regular by [HX13, 2.2], since $S$ is a smooth curve. Since the coefficients of $G$ are sufficiently small, $(S, D_S + G|_S)$ is strongly $F$-regular. Therefore we get the following surjection

$$F^e_* \mathcal{O}_S((1 - p^e)(K_S + D_S + G|_S)) \twoheadrightarrow \mathcal{O}_S$$

for all $e \gg 0$ and sufficiently divisible.

We have the following commutative diagram

$$
\begin{array}{ccc}
F^e_* \mathcal{O}_Y((1 - p^e)(K_Y + S + G)) & \longrightarrow & F^e_* \mathcal{O}_S((1 - p^e)(K_S + D_S + G|_S)) \\
\downarrow & & \downarrow \\
\mathcal{O}_Y & \longrightarrow & \mathcal{O}_S
\end{array}
$$

To see the surjectivity of the top arrow note that since $F^e_*$ is exact it suffices to show that $(1 - p^e)(K_Y + S + G)|_S = (1 - p^e)(K_S + D_S + G|_S)$, and since $(1 - p^e)(K_Y + S + G)$ and $(1 - p^e)(K_S + D_S + G|_S)$ are Cartier for $e \gg 0$, it suffices to show that this equality holds at codimension 1 points of $S$, but this is clear since $(K_Y + S + G) = K_S + D_S + G|_S$. Since the ring $\mathcal{O}_Y$ is local, the surjectivity of the second vertical map (along with Nakayama’s lemma) implies the surjectivity of the first vertical map, i.e.,

$$
F^e_* \mathcal{O}_Y((1 - p^e)(K_Y + S + G)) \twoheadrightarrow \mathcal{O}_Y
$$

is surjective.

Since the map (2.11) factors through $F^e_* \mathcal{O}_Y((1 - p^e)K_Y)$, we get the following surjectivity

$$
F^e_* \mathcal{O}_Y((1 - p^e)K_Y) \xrightarrow{\psi_e} \mathcal{O}_Y.
$$

Let $s$ be a pre-image of 1 under $\psi_e$. Then we get the following splitting of $\psi_e$

$$
\begin{array}{ccc}
\mathcal{O}_Y & \longrightarrow & F^e_* \mathcal{O}_Y((1 - p^e)K_Y) \\
\longrightarrow & \psi_e & \downarrow \\
& \mathcal{O}_Y
\end{array}
$$

Twisting (2.13) by $\mathcal{O}_Y(-S)$ and taking reflexive hull we get that the following splitting

$$
\begin{array}{ccc}
\mathcal{O}_Y(-S) & \longrightarrow & F^e_* \mathcal{O}_Y((1 - p^e)K_Y - p^eS) \\
\longrightarrow & \psi_e & \downarrow \\
& \mathcal{O}_Y(-S)
\end{array}
$$

In particular the morphism

$$F^e_* \mathcal{O}_Y((1 - p^e)K_Y - p^eS) \longrightarrow \mathcal{O}_Y(-S)$$

is surjective and Claim (2.3) follows.

\[ \square \]

**Theorem 2.6.** Let $(X, \Delta)$ be a $\mathbb{Q}$-factorial 3-fold log canonical pair such that $X$ has KLT singularities. If $W$ is a minimal log canonical center of $(X, \Delta)$, then $W$ is normal.
Proof. Since $X$ is $\mathbb{Q}$-factorial and all log canonical centers of $(X, \Delta)$ are contained in $\Delta$, $(X, (1-\epsilon)\Delta)$ is KLT for any $0 < \epsilon < 1$. Then by Reid’s Tie Breaking trick (see [Cor07, 8.7.1]) we may assume that $W$ is the unique log canonical center of $(X, \Delta)$ with a unique exceptional divisor over $X$ of discrepancy $-1$. There are two cases depending on the codimension of $W$.

**Case I:** $\text{codim}_X(W) = 1$. Since $X$ is $\mathbb{Q}$-factorial, $(X, W)$ is log canonical. By adjunction $(K_X + W)|_{W^n} = K_{W^n} + \text{Diff}_{W^n}$, where $W^n \to W$ is the normalization and $(W^n, \text{Diff}_{W^n})$ is KLT. Thus by [Har98] and [HX13, 3.1], $(W^n, \text{Diff}_{W^n})$ is strongly $F$-regular in characteristic $p > 5$. Then $W^n = W$, i.e., $W$ is normal by [HX13, 4.1] or [Das13, 4.1].

**Case II:** $\text{codim}_X(W) = 2$. Let $f : (Y, S + B) \to (X, \Delta)$ be a divisorial extraction such that

$$K_Y + S + B = f^*(K_X + \Delta)$$

where $S$ is the exceptional divisor over $W$.

$(Y, S + B)$ is PLT with $S$ an irreducible PLT center. Since $Y$ is $\mathbb{Q}$-factorial, $(Y, S)$ is also PLT. By adjunction we have $(K_Y + S)|_{S^n} = K_{S^n} + \text{Diff}_{S^n}$, where $S^n \to S$ is the normalization. Then $(S^n, \text{Diff}_{S^n})$ is KLT. Hence by [HX13, 3.1], $(S^n, \text{Diff}_{S^n})$ is strongly $F$-regular in characteristic $p > 5$, and so $S^n = S$, i.e. $S$ is normal by [HX13, 4.1] or [Das13, 4.1, 5.4].

Consider the following exact sequence

$$0 \to \mathcal{O}_Y(\mathcal{O}_Y(-S)) \to \mathcal{O}_Y \to \mathcal{O}_S \to 0$$

By Theorem 2.5 we have $R^1f_*\mathcal{O}_Y(-S) = 0$. Thus we get the following exact sequence

$$0 \to f_*\mathcal{O}_Y(-S) \to f_*\mathcal{O}_Y \to f_*\mathcal{O}_S \to 0.$$
where $\nu : W^m \to W$ is the normalization morphism.

Hence $O_W = \nu_* O_{W^m}$, i.e. $W$ is normal. □

3. **Adjunction Formula on 3-folds**

In this section we will prove an adjunction formula on 3-folds and in characteristic $p > 5$. To start with we will need the following definitions and results.

### 3.1. DCC sets.

We say that a set $I$ of real numbers satisfies the *descending chain condition* or DCC, if it does not contain any infinite strictly decreasing sequence. For example,

$$I = \{ \frac{r-1}{r} : r \in \mathbb{N} \}$$

satisfies the DCC.

Let $I \subseteq [0, 1]$. We define

$$I_+ := \{ j \in [0, 1] : j = \sum_{p=1}^{l} i_p \text{ for some } i_1, i_2, \ldots, i_l \in I \}$$

and

$$D(I) := \{ a \leq 1 : a = \frac{m-1+f}{m}, m \in \mathbb{N}, f \in I_+ \}.$$

**Lemma 3.1.** [MP04, 4.4] Let $I \subseteq [0, 1]$. Then

1. $D(D(I)) = D(I) \cup \{1\}$.
2. $I$ satisfies DCC if and only if $\bar{I}$ satisfies the DCC, where $\bar{I}$ is the closure of $I$.
3. $I$ satisfies DCC if and only if $D(I)$ satisfies the DCC.

**Lemma 3.2.** [CGS14, Lemma 2.3][MP04, Lemma 4.3][HMX14, Lemma 4.1] Let $(X, \Delta \geq 0)$ be a log canonical pair such that the coefficients of $\Delta$ belong to a set $I \subseteq [0, 1]$. Let $S$ be a normal irreducible component of $[\Delta]$ and $\Theta \geq 0$ be the $\mathbb{Q}$-divisor on $S$ defined by adjunction:

$$(K_X + \Delta)|_S = K_S + \Theta.$$
Then, the coefficients of $\Theta$ belong to $D(I)$.

3.2. **Divisorial parts and Moduli parts.** Let $f : X \to Z$ be a surjective proper morphism between two normal varieties and $K_X + D \sim_{Q} f^*L$, where $D$ is a boundary divisor. Let $(X, D)$ be LC near the generic fiber of $f$, i.e., $(f^{-1}U, D|_{f^{-1}U})$ is LC for some Zariski dense open subset $U \subseteq Z$. Then we define two divisors $D_{\text{div}}$ and $D_{\text{mod}}$ on $Z$ in the following way:

$$D_{\text{div}} = \sum (1 - c_Q)Q,$$

where $Q \subseteq Z$ are prime Weil divisors of $Z$,

$$c_Q = \sup \{ c \in \mathbb{R} : (X, D + cf^*Q) \text{ is LC over the generic point } \eta_Q \text{ of } Q \}$$

and

$$D_{\text{mod}} = L - K_Z - D_{\text{div}},$$

so that $K_X + D \sim_{Q} f^*(K_Z + D_{\text{div}} + D_{\text{mod}})$.

**Remark 3.3.** Observe that $D_{\text{div}}$ is a fixed divisor on $Z$, called the **Divisorial part** and $D_{\text{mod}}$ is a $\mathbb{Q}$-linear equivalence class on $Z$, called the **Moduli part**. For other properties of $D_{\text{div}}$ and $D_{\text{mod}}$ see [PS09, Section 7] and [Amb99, Section 3].

Let $\mathcal{M}_{0,n}$ be the moduli space of $n$-pointed stable curves of genus 0, $f_{0,n} : \mathcal{U}_{0,n} \to \mathcal{M}_{0,n}$ the universal family, and $\mathcal{P}_1, \mathcal{P}_2, \cdots, \mathcal{P}_n$, the sections of $f_{0,n}$ which correspond to the marked points (see [Kee92] and [Knu83]). Let $d_j (j = 1, 2, \cdots, n)$ be rational numbers such that $0 < d_j \leq 1$ for all $j$, $\sum j d_j = 2$ and $\mathcal{D} = \sum j d_j \mathcal{P}_j$.

**Lemma 3.4.** (1) There exists a smooth projective variety $U^*_{0,n}$, a $\mathbb{P}^1$-bundle $g_{0,n} : U^*_{0,n} \to \mathcal{M}_{0,n}$, and a sequence of blowups with smooth centers

$$
\mathcal{U}_{0,n} = U^{(1)} \xrightarrow{\sigma_2} U^{(2)} \xrightarrow{\sigma_3} \cdots \xrightarrow{\sigma_{n-2}} U^{(n-2)} = U^*_{0,n}.
$$

(2) Let $\sigma : \mathcal{U}_{0,n} \to U^*_{0,n}$ and $g_{0,n} : U^*_{0,n} \to \mathcal{M}_{0,n}$ be the induced morphisms, and $\mathcal{D}^* = \sigma_* \mathcal{D}$. Then $K_{U_0,n} + \mathcal{D} - \sigma^*(K_{U^*_{0,n}} + \mathcal{D}^*)$ is effective.

(3) There exists a semi-ample $\mathbb{Q}$-divisor $\mathcal{L}$ on $\mathcal{M}_{0,n}$ such that

$$K_{U^*_{0,n}} + \mathcal{D}^* \sim_{Q} g_{0,n}^*(K_{\mathcal{M}_{0,n}} + \mathcal{L}).$$

**Proof.** The proof in [Kaw97, Theorem 2] works in positive characteristic without any change (see also [CTX13, 6.7], [PS09, 8.5] and [KMM94, Section 3]).

**Lemma 3.5 (Stable Reduction Lemma).** Let $B$ be a smooth curve and $f : X \to B$, a flat family of rational curves such that the general fiber is isomorphic to $\mathbb{P}^1$, and a unique singular fiber $X_0$ over $0 \in B$. Also assume that $f|_{X^*} : (X^* = X \setminus X_0; \mathcal{P}_1, \mathcal{P}_2, \cdots, \mathcal{P}_n) \to B^* = B \setminus \{0\}$ is a flat family of $n$-pointed
stable rational curves sitting in the following commutative diagram\(^2\)

\[
\begin{array}{ccc}
X^* = B^* \times_{\overline{\mathcal{M}}_{0,n}} \overline{U}_{0,n} & \xrightarrow{f} & \overline{U}_{0,n} \\
\downarrow & & \downarrow \\
B^* & \xrightarrow{id_B} & \overline{\mathcal{M}}_{0,n}
\end{array}
\]

Then there exists a unique flat family \(\hat{f}: \hat{X} \to B\) of \(n\)-pointed stable rational curves satisfying the following commutative diagram

\[
\begin{array}{ccc}
X \xrightarrow{f} \hat{X} = B \times_{\overline{\mathcal{M}}_{0,n}} \overline{U}_{0,n} & \xrightarrow{j} & \overline{U}_{0,n} \\
\downarrow & & \downarrow \\
B & \xleftarrow{id_B} & \overline{\mathcal{M}}_{0,n}
\end{array}
\]

where the broken horizontal map is a birational map such that \(f^{-1}B^* \cong \hat{f}^{-1}B^*\).

**Proof.** Since \(\overline{\mathcal{M}}_{0,n}\) is a proper scheme, by the valuative criterion of properness any morphism \(B^* \to \overline{\mathcal{M}}_{0,n}\) extends uniquely to a morphism \(B \to \overline{\mathcal{M}}_{0,n}\). Now since \(\overline{\mathcal{M}}_{0,n}\) has a universal family \(\overline{U}_{0,n}\), the existence of \(\hat{f}: \hat{X} \to B\) follows by taking the fiber product. \(\square\)

**Theorem 3.6** (Canonical Bundle Formula). Let \(f: X \to Z\) be a proper surjective morphism, where \(X\) is a normal surface and \(Z\) is a smooth curve over an algebraically closed field \(k\) of \(\text{char}(k) > 0\). Also assume that \(Q = \sum_i Q_i\) is a divisor on \(Z\) such that \(f\) is smooth over \((Z - \text{Supp}(Q))\) with fibers isomorphic to \(\mathbb{P}^1\). Let \(D = \sum_j d_j P_j\) be a \(Q\)-divisor on \(X\), where \(d_j = 0\) is allowed, which satisfies the following conditions:

1. \((X, D \geq 0)\) is KLT.
2. \(D = D^h + D^v\), where \(D^h = \sum_{f(D_j) = Z} d_j D_j\) and \(D^v = \sum_{f(D_j) \neq Z} d_j D_j\).
   An irreducible component of \(D^h\) (resp. \(D^v\)) is called horizontal (resp. vertical) component.
3. \(\text{Char}(k) = p > \frac{2}{\delta}\), where \(\delta\) is the minimum non-zero coefficient of \(D^h\).
4. \(K_X + D \sim_{\mathbb{Q}} f^*(K_Z + M)\) for some \(\mathbb{Q}\)-Cartier divisor \(M\) on \(Z\).

Then there exist an effective \(\mathbb{Q}\)-divisor \(D_{\text{div}} \geq 0\) and a semi-ample \(\mathbb{Q}\)-divisor \(D_{\text{mod}} \geq 0\) on \(Z\) (as defined in 3.2) such that

\[
K_X + D \sim_{\mathbb{Q}} f^*(K_Z + D_{\text{div}} + D_{\text{mod}}).
\]

**Proof.** The sketch of the proof of this formula is given in [CTX13, 6.7]. We include a complete proof following the idea of the proof of [PS09, Theorem 8.1].

---

\(^2\)don’t we need to base change so that the \(P_i\) are sections?
First we reduce the problem to the case where all components of $D^h$ are sections. Let $D_{i_0}$ be a horizontal component of $D$ and $Z' \to D_{i_0}$ be the normalization of $D_{i_0}$. Then $\nu : Z' \to Z$ is a finite surjective morphism of smooth curves. Let $X'$ be the normalization of the component of $X \times_Z Z'$ dominating $Z$.

\[
\begin{array}{ccc}
X & \xleftarrow{\nu'} & X' \\
f \downarrow & & f' \downarrow \\
Z & \xleftarrow{\nu} & Z'
\end{array}
\]

Let $k = \deg(\nu : Z' \to Z)$ and $l$ be a general fiber of $f$. Then

\[
k = D_i \cdot l \leq \frac{1}{d_i}(D \cdot l) = \frac{1}{d_i}(-K_X \cdot l) = \frac{2}{d_i} \leq \frac{2}{\delta} < \text{Char } (k).
\]

Therefore $\nu : Z' \to Z$ is a separable morphism.

Let $D'$ be the log pullback of $D$ under $\nu'$, i.e.,

\[
K_{X'} + D' = \nu'*(K_X + D).
\]

More precisely we have (by [Kol92, 20.2])

\[
D' = \sum_{i,j} d'_{ij} D'_{ij}, \quad \nu'(D'_{ij}) = D_i, \quad d'_{ij} = 1 - (1 - d_i)e_{ij},
\]

where $e_{ij}$'s are the ramification indices along the $D'_{ij}$'s.

By construction $X$ dominates $Z$. Also, since $\nu$ is etale over a dense open subset of $Z$, say, $\nu^{-1}U \to U$, and etale morphisms are stable under base change, $(f' \circ \nu)^{-1}U \to f^{-1}U$ is etale. Thus the ramification locus $\Lambda$ of $\nu'$ does not contain any horizontal divisor of $f'$, i.e., $f'(-\Lambda) \neq Z'$. Therefore $D'$ is boundary near the generic fiber ($\cong \mathbb{P}^1$) of $f'$, i.e., $D'^h$ is effective. We observe that the coefficients of $D'^h$ can be computed by intersecting with a general fiber of $f' : X' \to Z'$, hence they are equal to the coefficients of $D^h \subseteq X$. Thus the condition $p > \frac{2}{\delta}$ remains true for $D'$ on $X'$.

After finitely many such base changes let $g : \tilde{X} \to \tilde{Z}$ be a family such that all of the horizontal components of $D_{\tilde{X}}$ are sections of $g$, where $D_{\tilde{X}}$ is the log pullback of $D$, i.e., $K_{\tilde{X}} + D_{\tilde{X}} = \psi^*(K_X + D)$.

\[
\begin{array}{ccc}
X & \xleftarrow{\psi} & \tilde{X} \\
f \downarrow & & g \downarrow \\
Z & \xleftarrow{\psi_0} & \tilde{Z}
\end{array}
\]
By Lemma 3.5, we get a family of \( n \)-pointed stable rational curves \( \tilde{X} = \tilde{Z} \times_{\mathcal{M}_{0,n}} \bar{U}_{0,n} \to \tilde{Z} \). Let \( X' \) be the common resolution of \( \tilde{X} \) and \( \hat{X} \). Let \( \hat{X} = \hat{Z} \times_{\mathcal{M}_{0,n}} \mathcal{U}_{0,n}^* \). By the universal property of fiber product there exists a morphism \( \mu : X' \to \hat{X} \). Since \( X', \tilde{X} \) and \( \hat{X} \) are all isomorphic \( \mathbb{P}^1 \)-bundles over a dense open subset \( U \subseteq \tilde{Z} \), \( \mu : X' \to \hat{X} \) is birational.

(3.7)

\[
\begin{align*}
X & \xrightarrow{\psi} \tilde{X} & \xrightarrow{\hat{f}} \hat{X} & \xrightarrow{\sigma} \mathcal{U}_{0,n}^* \\
\downarrow f & \downarrow \psi_0 & \downarrow \hat{f} & \downarrow f_0, n \\
Z & \xrightarrow{g_0, n} \mathcal{M}_{0,n}
\end{align*}
\]

Let \( D' \) and \( \hat{D} \) be \( \mathbb{Q} \)-divisors on \( \tilde{X} \) and \( \hat{X} \) respectively, defined by

(3.8)

\[
K_{X'} + D' = \pi^*(K_X + D).
\]

and

\[
K_{\tilde{X}} + \hat{D} = \mu^*(K_{\hat{X}} + \hat{D}).
\]

Since \( K_{X'} + D' \) is a pullback from the base \( \tilde{Z} \) (by (3.7)), by the Negativity lemma we get

(3.9)

\[
K_{X'} + D' = \mu^*(K_{\tilde{X}} + \hat{D}).
\]

Since the definition of the divisorial part of the adjunction does not depend on the birational modification of the family (see [PS09, Remark 7.3(ii)] or [Amb99, Remark 3.1]), we will define it with respect to \( \hat{f} : \hat{X} \to \tilde{Z} \). First we will show that the \( \mathbb{Q} \)-divisor \( \hat{D}_{\text{mod}} \) on \( \tilde{Z} \) is semi-ample.

Since \( \hat{\phi} \) is finite and \( \mathcal{D}^* \) is horizontal it follows that \( \hat{\phi}^*(\mathcal{D}^*) \) is horizontal too. Since \( \hat{D}^h \) is also horizontal one sees that

\[
\hat{D}^h = \hat{\phi}^*(\mathcal{D}^*).
\]

From the construction of \( \sigma : \mathcal{U}_{0,n} \to \mathcal{U}_{0,n}^* \) we see that \( (F, \mathcal{D}^*|_F) \) is log canonical for any fiber \( F \) of \( g_{0,n} : \mathcal{U}_{0,n}^* \to \mathcal{M}_{0,n} \). Since the fibers of \( \hat{f} : \hat{X} \to \tilde{Z} \) are isomorphic to the fibers of \( g_{0,n} : \hat{U}_{0,n}^* \to \mathcal{M}_{0,n} \), \( (\hat{F}, \hat{D}^h|_{\hat{F}}) \) is also log canonical, where \( \hat{F} \) is a fiber of \( \hat{f} \). Finally, since \( \hat{X} \) is a surface, by inversion of adjunction \( (\hat{X}, \hat{F} + \hat{D}^h) \) is log canonical near \( \hat{F} \). Thus, since the fibers are reduced, we get

(3.10)

\[
\hat{D}^v = f^* \hat{D}_{\text{div}}
\]
and, by definition of $\hat{D}_{mod}$ we have
\begin{equation}
K_{\hat{X}} + \hat{D}^h \sim_Q \hat{f}^*(K_{\tilde{Z}} + \hat{D}_{mod}).
\end{equation}

By (3.11), Lemma 3.4 and [Liu02, Chapter 6, Theorem 4.9 (b) and Example 3.28] we get
\begin{equation}
K_{\hat{X}} + \hat{D}^h - \hat{f}^*(K_{\tilde{Z}} + \phi^*_0 L) = K_{\hat{X}} - \hat{f}^* \phi^*(\mathcal{D}) \sim_Q 0.
\end{equation}

Since $\hat{f}$ has connected fibers, by (3.11) and (3.12) and the projection formula for locally free sheaves, we get
\begin{equation}
\hat{D}_{mod} \sim_Q \phi^*_0 L
\end{equation}
i.e., $\hat{D}_{mod}$ is semi-ample.

Now, since $\psi_0 : \tilde{Z} \to Z$ is a composition of finite morphisms of degree strictly less than $\text{Char}(k)$, by [Kol13, Corollary 2.43] and [Amb99, Theorem 3.2] (also see [CTX13, 6.6]) we get
\begin{equation}
K_{\tilde{Z}} + \hat{D}_{div} \sim_Q \psi^*(K_Z + D_{div}).
\end{equation}
Therefore
\begin{equation}
\psi^* D_{mod} \sim_Q \hat{D}_{mod}
\end{equation}
Since $Z$ and $\tilde{Z}$ are both smooth curves, $D_{mod}$ is semi-ample.

\begin{theorem}
Let $(X, D \geq 0)$ be a $\mathbb{Q}$-factorial 3-fold log canonical pair such that the coefficients of $D$ are contained in a DCC set $I \subseteq [0, 1]$. Let $W$ be a minimal log canonical center of $(X, D)$, and codimension of $W$ is 2. Also assume that $X$ has KLT singularities and $\text{Char}(k) > \max\{5, \frac{2}{3}\}$, where $\delta$ is the non-zero minimum of the set $D(I)$ (defined in 3.1). Then the following hold:
\begin{enumerate}
\item $W$ is normal.
\item There exists effective $\mathbb{Q}$-divisors $D_W$ and $M_W$ on $W$ such that $(K_X + D)|_W \sim_Q K_W + D_W + M_W$. Moreover, if $D = D' + D''$ with $D'$ (resp. $D''$) the sum of all irreducible components which contain (resp. do not contain) $W$, then $M_W$ is determined only by the pair $(X, D')$.
\item There exists an effective $\mathbb{Q}$-divisor $M'_W$ such that $M'_W \sim_Q M_W$ and the pair $(W, D_W + M'_W)$ is KLT.
\end{enumerate}
\end{theorem}

\begin{proof}
Normality of $W$ follows from Theorem 2.6.

Since $X$ is $\mathbb{Q}$-Cartier, $(K_X + D)|_W = (K_X + D' + D'')|_W = (K_X + D')|_W + D''|_W$. Thus we may assume that all components of $D$ contain $W$. Since $W$ is a minimal log canonical center of $(X, D)$ and $\text{codim}_X W = 2$, it does not
intersect any other LC center of codimension $\geq 2$, by Lemma 2.1. Thus by shrinking $X$ (removing closed subsets of codimension $\geq 2$ which do not intersect $W$) if necessary we may assume that $W$ is the unique log canonical center of codimension $\geq 2$ of $(X, D)$.

Let $f : (X', D') \to (X, D)$ be a $\mathbb{Q}$-factorial DLT model over $(X, D)$ such that
\begin{equation}
K_{X'} + D' = f^*(K_X + D).
\end{equation}
Such $f$ exists by [KK10, 3.1] and [Bir13].

Note that, since $X$ is $\mathbb{Q}$-factorial, the exceptional locus of $f$ supports an effective anti-ample divisor. In particular all positive dimensional fibers of $f$ are contained in the support of $\lfloor D' \rfloor$.

Let $E$ be an exceptional divisor dominating $W$. Then $E$ is normal by Proposition 2.2. Write $D' = E + \sum d_i f_i^{-1} D_i$. By adjunction we have
\begin{equation}
K_E + D'_E = (K_{X'} + D')|_E = f^*((K_X + D)|_W)
\end{equation}
and $(E, D'_E)$ is DLT, by Proposition 2.2 and the coefficients of $D'_E$ are in the set $D(I)$ by Lemma 3.2.

By Theorem 3.6, there exist $\mathbb{Q}$-divisors $D_W \geq 0$ and $M_W \geq 0$ on $W$ such that
\begin{equation}
K_E + D'_E \sim_{\mathbb{Q}} f|_E^*(K_W + D_W + M_W).
\end{equation}
Since $f|_E : E \to W$ has connected fibers, from (3.16), (3.17) and the projection formula for locally free sheaves, we get
\begin{equation}
(K_X + D)|_W \sim_{\mathbb{Q}} K_W + D_W + M_W.
\end{equation}
Lemma 3.8 given below shows that $D_W$ is independent of the choice of the exceptional divisor $f$ dominating $W$.

From the definition of $D_W$ we see that $D_W \geq 0$, since $D'_E \geq 0$. Also, since $D_W$ is independent of the birational modifications (by [PS09, Remark 7.3(ii)]) and $W$ is a minimal LC center, by taking a log resolution of $(X', D')$ and working on the strict transform of $E$, we see that the coefficients of $D_W$ are strictly less than 1. Thus $\lfloor D_W \rfloor = 0$.

Since $M_W$ is semi-ample and $W$ is a smooth curve, either $M_W = 0$ or $M_W$ is ample. In the later case by Bertini’s theorem there exists an effective $\mathbb{Q}$-divisor $M'_W \sim_{\mathbb{Q}} M_W$ such that $\lfloor M'_W \rfloor = 0$ and $\text{Supp}(M'_W) \cap \text{Supp}(D_W) = \emptyset$. Hence $(W, D_W + M'_W)$ is KLT.
Lemma 3.8. With the same hypothesis as in Theorem 3.7, the divisor $D_W = D_{\text{div}}$ on $W$ is independent of the choice of the exceptional divisors dominating $W$.

Proof. Let $E_1$ and $E_2$ be two exceptional divisors of $f$ dominating $W$ such that
\begin{equation}
K_{X'} + E_1 + E_2 + \Delta' = f^*(K_X + D),
\end{equation}
where $f : X' \to X$ is the DLT model as above and $D' = E_1 + E_2 + \Delta'$.

By adjunction on $E_1$ we get
\begin{equation}
K_{E_1} + C + \Delta'_{E_1} = f^*((K_X + D)|_W),
\end{equation}
where $C$ is an irreducible component of $E_1 \cap E_2$.

Adjunction on $C$ gives
\begin{equation}
K_C + \Delta'_C = f^*((K_W + D)|_W).
\end{equation}

Let $Q$ be a point on $W$, and $t = \text{lct}(E_1, C + \Delta'_{E_1}; f^*Q)$ and $s = \text{lct}(C, \Delta'_C; f^*Q|_C)$. Since $C$ is an irreducible component of $E_1 \cap E_2$ dominating $W$, it is enough to show that $t = s$. By adjunction, $t \leq s$. So by contradiction assume that $t < s$.

Since $(E_1, C + \Delta'_{E_1})$ DLT by Proposition 2.2, $(E_1, C + \Delta'_{E_1} + t'f^*Q)$ is LC outside of $f^{-1}Q$ for any $t' > t$. Thus all NLC centers of $(E_1, C + \Delta'_{E_1} + t'f^*Q)$ appear along $f^{-1}Q$.

The general fiber of $f|_{E_1} : E_1 \to W$ is isomorphic to $\mathbb{P}^1$. Thus degree($(C + \Delta'_{E_1})|_{\mathbb{P}^1}) = 2$ by (3.20). There are two cases depending on whether $C$ intersects the general fiber with degree 1 or 2.

**Case I:** $C$ intersects the general fiber with degree 1. Then there exists a horizontal component $C'$ of $\Delta'_{E_1}$. Let $H$ be an ample divisor on $E_1$, and $F_\eta$, the generic fiber of $f|_{E_1} : E_1 \to W$. Choose $\lambda > 0$ such that
\[(H - \lambda C') \cdot F_\eta = 0.\]
Then $(H - \lambda C')|_{F_\eta} \sim 0$. Thus by [Cor07, 8.3.4], $H \sim \lambda C' - \sum \lambda_i F_i$, where the $F_i$'s are irreducible components of some fibers of $f$. By adding pullback of some appropriate divisors from the base to $\lambda C' - \sum \lambda_i F_i$, we may assume that $\lambda_i > 0$ for all $i$ and $\lambda C' - \sum \lambda_i F_i$ is $f$-ample.

Assume that there exists a point $P \in f^{-1}Q$ but $P \notin C$ such that $(E_1, C + \Delta'_{E_1} + (t + \epsilon)f^*Q)$ is not LC at $P$, where $0 < \epsilon \ll 1$ such that $t + \epsilon < s$. Then
by choosing $0 < \lambda, \lambda_i \ll 1$ we can assume that $(C + \Delta'_{E_1} - \lambda C' + \sum \lambda_i F_i) \geq 0$, $(E_1, C + \Delta'_{E_1} - \lambda C' + \sum \lambda_i F_i + (t + \epsilon)f^*Q)$ still not LC at $P$, and

$$
(3.22) \quad -\left(K_{E_1} + C + \Delta'_{E_1} - \lambda C' + \sum \lambda_i F_i\right) = -f^*((K_X + D)|_W) + \left(\lambda C' - \sum \lambda_i F_i\right)
$$

is $f$-ample.

Then by [Bir13, 8.3], NKLT$(E_1, C + \Delta'_{E_1} - \lambda C' + \sum \lambda_i F_i + (t + \epsilon)f^*Q) \cap f^{-1}Q$ is connected. Let $R \in C \cap f^{-1}Q$. Then there exists a chain of curves $G_i$'s connecting $R$ and $P$, and contained in NKLT$(E_1, C + \Delta'_{E_1} - \lambda C' + \sum \lambda_i F_i + (t + \epsilon)f^*Q) \cap f^{-1}Q$.

Now NKLT$(E_1, C + \Delta'_{E_1} - \lambda C' + \sum \lambda_i F_i + (t + \epsilon)f^*Q) \subseteq$ NKLT$(E_1, C + \Delta'_{E_1} + \sum \lambda_i F_i + (t + \epsilon)f^*Q)$. Since we are only concentrating on the NKLT centers along $f^{-1}Q$, we may assume that $F_i$'s are all contained in $f^{-1}Q$. Then by choosing $0 < \lambda_i \ll 1$ for all $i$, such that $t + \epsilon = t + \epsilon + \max(\lambda_i) < s$, we see that NKLT$(E_1, C + \Delta'_{E_1} + \sum \lambda_i F_i + (t + \epsilon)f^*Q) \subseteq$ NKLT$(E_1, C + \Delta'_{E_1} + (t + \epsilon)f^*Q)$. Thus the curves $G_i$'s are contained in the NKLT$(E_1, C + \Delta'_{E_1} + (t + \epsilon)f^*Q)$. Hence $G_i$'s are contained in NLC$(E_1, C + \Delta'_{E_1} + sf^*Q)$. This implies that $(E_1, C + \Delta'_{E_1} + sf^*Q)$ is not LC at $R \in C$. Then by inversion of adjunction we get a contradiction to the fact that $(C, \Delta'_{E_1} + sf^*Q|_C)$ is LC.

**Case II:** $C$ intersects the general fiber with degree 2. In this case $E_1 \cap E_2 = C$ and $\Delta'_{E_1} = \Delta'_{E_2} = 0$. Since $D \neq 0$ and every component of $D$ contains $W$, one of the $E_i$'s, say $E_2 = f_s^{-1}D_1$, where $D_1$ is an irreducible component of $D$. Thus in this case the exceptional divisors of $f$ do not intersect each other. Since $X$ is $\mathbb{Q}$-factorial, the exceptional locus Ex$(f)$ of $f : X' \to X$ supports an effective anti-ample divisor and hence Ex$(f) \cap f^{-1}(w)$ is connected for all $w \in W$. Thus $f$ has a unique exceptional divisor in this case and we are done.

□

References


ON THE ADJUNCTION FORMULA ON 3-FOLDS IN CHARACTERISTIC $p$


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