Math 2250-1
Newton's law of cooling application - Maple Project 1
Solutions

I began by downloading "project1.mws" from our Maple page, and created this solution document by modifying it. (I did a lot of cutting, pasting, deleting, in addition to adding my own text and math.)

....

Exercise 1. *Using the integration factor algorithm we've learned for solving first order linear DEs, and working by hand, show that the general solution to the DE (1) above is given just as the book claims in equation 4, page 57. In other words, that the solutions are*

$$u(t) = a_0 + c_0\, e^{(-k\, t)} + c_1 \cos(\omega\, t) + c_2 \sin(\omega\, t).$$
(4)

Solution: We start with the DE

$$\frac{du}{dt} + k\, u = k\,(a_0 + a_1 \cos(\omega\, t) + b_1 \sin(\omega\, t)).$$
(3)

The integrating factor for this linear DE is $e^{(k\, t)}$, so we write the equivalent equation

$$e^{(k\, t)}\left(\frac{du}{dt} + k\, u\right) = e^{(k\, t)}\, k\,(a_0 + a_1 \cos(\omega\, t) + b_1 \sin(\omega\, t))$$
(3b)

Since the left side is the derivative of $e^{(k\, t)}\, u(t)$, we antidifferentiate to get

$$e^{(k\, t)}\, u(t) = \int e^{(k\, t)}\, k\,(a_0 + a_1 \cos(\omega\, t) + b_1 \sin(\omega\, t))\, dt$$

$$= a_0\, k \int e^{(k\, t)}\, dt + a_1\, k \int e^{(k\, t)} \cos(\omega\, t)\, dt + b_1\, k \int e^{(k\, t)} \sin(\omega\, t)\, dt$$
(3c)

Antidifferentiating, and using back cover table entries 49, 50 for the last two integrals we get

$$e^{(k\, t)}\, u(t) =$$

$$a_0\, e^{(k\, t)} + a_1\, k\left(\frac{k\, e^{(k\, t)} \cos(\omega\, t)}{k^2 + \omega^2} + \frac{\omega\, e^{(k\, t)} \sin(\omega\, t)}{k^2 + \omega^2}\right) + b_1\, k\left(-\frac{\omega\, e^{(k\, t)} \cos(\omega\, t)}{k^2 + \omega^2} + \frac{k\, e^{(k\, t)} \sin(\omega\, t)}{k^2 + \omega^2}\right) + C$$

Dividing both sides of this equation by yields

$$u(t) = a_0 + \frac{a_1\, k\,(k \cos(\omega\, t) + \omega \sin(\omega\, t))}{k^2 + \omega^2} + \frac{b_1\, k\,(-\omega \cos(\omega\, t) + k \sin(\omega\, t))}{k^2 + \omega^2} + C\, e^{(-k\, t)}$$
(3d)

Collecting coefficients for each function on the right side yields

$$u(t) = a_0 + \frac{\cos(\omega\, t)\, k\,(a_1\, k - b_1\, \omega)}{k^2 + \omega^2} + \frac{\sin(\omega\, t)\, k\,(a_1\, \omega + b_1\, k)}{k^2 + \omega^2} + C\, e^{(-kt)}.$$

$$= a_0 + c_0\, e^{(-k\, t)} + c_1 \cos(\omega\, t) + c_2 \sin(\omega\, t)$$

with

$$c_1 = \frac{k^2\, a_1 - k\, b_1\, \omega}{k^2 + \omega^2}, \quad c_2 = \frac{k\, a_1\, \omega + b_1\, k^2}{k^2 + \omega^2}$$

which agrees with the text (which calls $c_2 = d_1$.) The text calls our constant of integration $C = c_0$, and you can solve for it in terms of the intial value of u, by setting t=0 in the equation for u(t), i.e
$u_0 = a_0 + c_0 + c_1$, so

$$c_0 = u_0 - a_0 - c_1$$

which agrees with the text.

....

Exercise 2. Redo the computation for Exercise 1 using Maple to compute all the integrals you need. Your work here should all be done in the Maple worksheet: The Maple commands and output should be shown, along with any text explanation or command comments you find necessary.

Solution: The only integral we need to have Maple derive is the one we evaluated in (3c), i.e.
```
> k*int(exp(k*t)*(a[1]*cos(omega*t)+b[1]*sin(omega*t)),t);
```

$$k\left(a_1\left(\frac{k\,e^{(kt)}\cos(\omega t)}{k^2+\omega^2}+\frac{\omega\,e^{(kt)}\sin(\omega t)}{k^2+\omega^2}\right)+b_1\left(-\frac{\omega\,e^{(kt)}\cos(\omega t)}{k^2+\omega^2}+\frac{k\,e^{(kt)}\sin(\omega t)}{k^2+\omega^2}\right)\right)$$

which is what is displayed between (3c) and (3d) along with the constant C of integration. All the other work may be done as indicated in Exercise 1. (Your only grade on #2 will be whether you used Maple to compute this integral or its equivalent.)

....

Notice that the exponential part of the solution to (4) decays to zero (exponentially) as $t \to \infty$, and that what remains in the limit is also a solution (i.e. when the integration constant $c_0=0$). We call this the steady periodic (sp) solution, and write

$$u_{sp}(t) = a_0 + c_1\,\cos(\omega t) + c_2\,\sin(\omega t).$$

....

Exercise 3. Use Maple to crunch the numbers and show that with the Athens Georgia temperature data (i.e. the ambient temperature A(t) given by equation (2), and insulation constant

$$k = .2$$

we get

$$u(t) = 80 + (e^{(-.2\,t)})(u_0 - 82.3351) + 2.3351\,\cos\left(\frac{1\,\pi\,t}{12}\right) - 5.6036\,\sin\left(\frac{1\,\pi\,t}{12}\right) \qquad (5)$$

and so also,

$$u_{sp}(t) = 80 + 2.3351\,\cos\left(\frac{1\,\pi\,t}{12}\right) - 5.6036\,\sin\left(\frac{1\,\pi\,t}{12}\right) \qquad (6)$$

Solution: We begin by defining u(t) from our work above. (You could also just use the Maple command dsolve!)
```
>
  c[1]:=k/(k^2+omega^2)*(a[1]*k-b[1]*omega);
  d[1]:=k/(k^2+omega^2)*(a[1]*omega+b[1]*k);   #use book's letter
```

```
   c[0]:=u[0]-a[0]-c[1];
   u:=t->a[0]+c[0]*exp(-k*t)+c[1]*cos(omega*t)+d[1]*sin(omega*t);
        #we just defined u(t) in terms of the various parameters
```

$$c_1 := \frac{k\,(a_1\,k - b_1\,\omega)}{k^2 + \omega^2}$$

$$d_1 := \frac{k\,(a_1\,\omega + b_1\,k)}{k^2 + \omega^2}$$

$$c_0 := u_0 - a_0 - \frac{k\,(a_1\,k - b_1\,\omega)}{k^2 + \omega^2}$$

$$u := t \rightarrow a_0 + c_0\,e^{(-k\,t)} + c_1\,\cos(\omega\,t) + d_1\,\sin(\omega\,t)$$

```
>
   Digits:=5;   #use just 5 significant digits for nicer output
   k:=.2;
   omega:=evalf(Pi/12); #decimal approximate
   a[0]:=80.;   #mean temperature
   a[1]:=-5.;
   b[1]:=evalf(-5*sqrt(3));   #this is all Athens Georgia data
```

$$Digits := 5$$

$$k := 0.2$$

$$\omega := 0.26180$$

$$a_0 := 80.$$

$$a_1 := \text{-}5.$$

$$b_1 := \text{-}8.6605$$

```
> u(t);   #This is equation (5)!
```

$$80. + (u_0 - 82.335)\,e^{(-0.2\,t)} + 2.3351\,\cos(0.26180\,t) - 5.6035\,\sin(0.26180\,t)$$

```
> usp:=t->80.+2.3351*cos(.26180*t)-5.6035*sin(.26180*t);
        #I just moused in the formula for u(t) and deleted the
   exponential
```

$$usp := t \rightarrow 80. + 2.3351\,\cos(0.26180\,t) - 5.6035\,\sin(0.26180\,t)$$

....

Exercise 4: *Use the cosine addition angle formula or Maple to show that the steady periodic solution given by equation (6) above, can be written in phase-amplitude form as*

$$u_{sp}(t) = 80 - 6.0707\,\cos\!\left(\frac{1\,\pi\,(t - 7.5082)}{12}\right)$$

so that the coolest and hottest indoor temperatures are around 7:30 a.m. and p.m., with smaller amplitude of around 6.07, vs. the outdoor amplitude of 10 degrees.
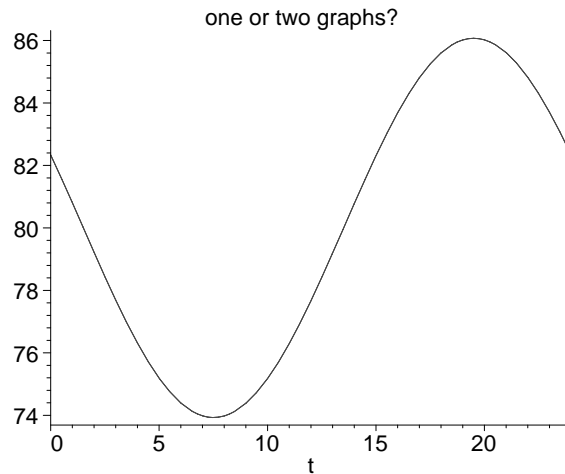
Solution: Clever students may figure out devious ways to have Maple verify these two functions are identical, for example you could show they have the same graphs:

```
> with(plots):
  plot1:=plot(80-6.0707*cos(1/12*Pi*(t-7.5082)),t=0..24,color=red):
  plot2:=plot(usp(t),t=0..24,color=black):
  display({plot1,plot2},title='one or two graphs?');
```


one or two graphs?

Or, you could work backwards using the cosine addition formula:
```
> pi:=evalf(Pi); #decimal for Pi
```
$$\pi := 3.1416$$
```
> 80-6.0707*(cos(pi*t/12)*cos(pi*7.5082/12)
     +sin(pi*t/12.)*sin(pi*7.5082/12));
         #expand the phase-amplitude expression!
```
$$80 + 2.3350 \cos(0.26180\,t) - 5.6037 \sin(0.26180\,t)$$

Notice this agrees with (6), so we're done!

But, what I'm hoping you do (even though you'll get full credit for the more devious ways), is use the handout on our Maple page for converting between the linear combo and phase amplitude forms of trig functions (or the text explanation). Then your argument will go something like:

Using the trigonometry handout on our Maple page, or the text, we want

$$80 + 2.3350 \cos(0.26180\,t) - 5.6037 \sin(0.26180\,t) \ = 80 - C \cos(0.26180\,t - \alpha)$$

So, being careful with minus signs,
$$-2.3350 \cos(0.26180\,t) + 5.6037 \sin(0.26180\,t) = C \cos(0.26180\,t - \alpha)$$
Thus
```
[ >
> A:=-2.3350;
  B:=5.6037;
  C:=sqrt(A^2+B^2);
  alpha:=arccos(A/C); #phase
  delta:=alpha/omega; #time delay
```
$$A := -2.3350$$
$$B := 5.6037$$

$$C := 6.0707$$
$$\alpha := 1.9656$$
$$\delta := 7.5080$$

Thus

$$\text{usp}(t) = 80 - 6.0707 \cos\left(\frac{\pi t}{12} - 1.9656\right)$$
$$= 80 - 6.0707 \cos\left(\frac{\pi (t - 7.5080)}{12}\right)$$

....

Now we're in Salt Lake City, it's winter, we're worried about pipes freezing after our heater breaks, and we're doing exercises L2.3 and L2.4 from Professor Gustafson's project! So, the setup is that our heater is turned off at midnight, t=0. *Perhaps we forgot to pay our natural gas bill. At this time, the indoor temperature is $u_0 = 74$ degrees. The outdoor ambient temperature is given as in equation (1), except that the 24-hour temperatures vary between 21 and 49 degrees, with the high and low at 3 p.m. and 3 a.m., respectively. We don't have a very well insulated house, its insulation constant is $k = .32$.*

....

Exercise 5: *From the information above deduce the constants in the formula for the ambient temperature A(t), and use Maple to find the solution to the initial value problem for the indoor temperature u(t).*

Solution: The average temperature is

```
> restart:   #clear old data
  Digits:=5:
> (21.+49.)/2;
```
$$35.000$$

and the amplitude of the temperature variation is
```
> 49-35;
```
$$14$$
```
> 
```
The min and max occur 3 and 15 hours after midnight, so we deduce the ambient temperature is given by
```
> omega:=evalf(Pi/12.);
```
$$\omega := 0.26180$$
```
> A:=t->35-14*cos(omega*(t-3));
```
$$A := t \rightarrow 35 - 14 \cos(\omega (t - 3))$$
```
> expand(A(t),trig);
```
$$35 - 9.8995 \cos(0.26180\, t) - 9.8995 \sin(0.26180\, t)$$
```
> unassign('omega');
```
Old definitions, copied in:
```
> c[1]:=k/(k^2+omega^2)*(a[1]*k-b[1]*omega);
  d[1]:=k/(k^2+omega^2)*(a[1]*omega+b[1]*k);   #use book's letter
  c[0]:=u[0]-a[0]-c[1];
  U:=t->a[0]+c[0]*exp(-k*t)+c[1]*cos(omega*t)+d[1]*sin(omega*t);
```

$$c_1 := \frac{k\,(a_1\,k - b_1\,\omega)}{k^2 + \omega^2}$$

$$d_1 := \frac{k\,(a_1\,\omega + b_1\,k)}{k^2 + \omega^2}$$

$$c_0 := u_0 - a_0 - \frac{k\,(a_1\,k - b_1\,\omega)}{k^2 + \omega^2}$$

$$U := t \to a_0 + c_0\,e^{(-k\,t)} + c_1\,\cos(\omega\,t) + d_1\,\sin(\omega\,t)$$

```
> omega:=evalf(Pi/12);
  k:=.32;
  a[0]:=35;
  a[1]:=-9.8995;
  b[1]:=-9.8995;
  u[0]:=74.;   #Salt Lake City constants
```

$$\omega := 0.26180$$

$$k := 0.32$$

$$a_0 := 35$$

$$a_1 := \text{-}9.8995$$

$$b_1 := \text{-}9.8995$$

$$u_0 := 74.$$

```
> U(t);   #it turns out it was a bad idea to use u(t)
           #for the function, and then write initial data
           #as u[0], Maple gets confused.  So I called the
           #temperature U(t) (capital U).
```

$$35 + 40.078\,e^{(-0.32\,t)} - 1.0785\,\cos(0.26180\,t) - 10.782\,\sin(0.26180\,t)$$

or, the shortcut:

```
> with(DEtools):
> deqtn:=diff(v(t),t)=k*(A(t)-v(t));
```

$$deqtn := \frac{d}{dt}\,v(t) = 11.20 - 4.48\,\cos(0.26180\,t - 0.78540) - 0.32\,v(t)$$

```
> evalf(dsolve([deqtn,v(0)=74.],v(t)));
```

$$v(t) = 35. - 8.3866\,\cos(0.26180\,t - 0.78540) - 6.8613\,\sin(0.26180\,t - 0.78540) + 40.078\,e^{(-0.32000\,t)}$$

```
> expand(%,trig);
```

$$v(t) = 35. - 1.0785\,\cos(0.26180\,t) - 10.782\,\sin(0.26180\,t) + 40.078\,e^{(-0.32000\,t)}$$

```
>
```

....

Exercise 6: *Create a Maple display which shows three plots: the Ambient temperature, the indoor temperature, and the steady-periodic indoor temperature.*
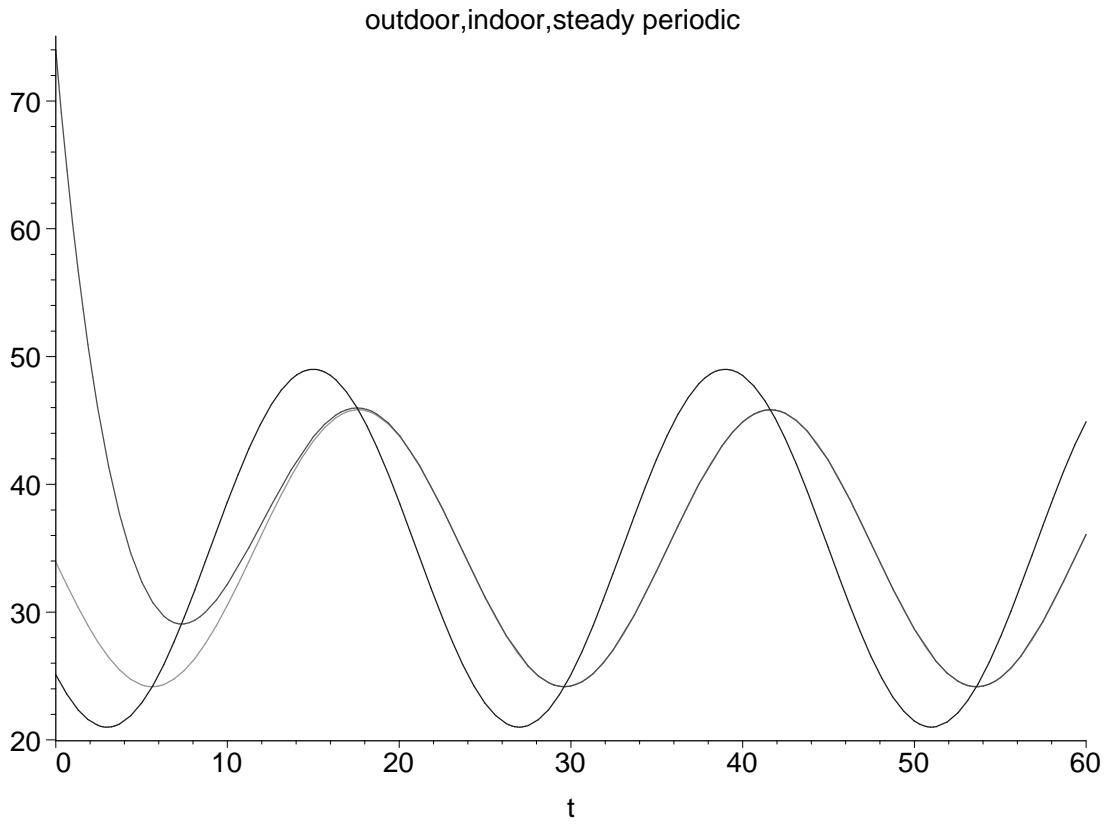
```
> with(plots):
> usp:=t->35-1.0785*cos(.26180*t)-10.782*sin(.26180*t);
```

$$usp := t \rightarrow 35 - 1.0785 \cos(0.26180\,t) - 10.782 \sin(0.26180\,t)$$

```
> plot1:=plot(U(t),t=0..60,color=red): #indoor
  plot2:=plot(A(t),t=0..60,color=blue): #exterior
  plot3:=plot(usp(t),t=0..60,color=green):#steady periodic indoor
  display({plot1,plot2,plot3},title='outdoor,indoor,steady
  periodic');
```


outdoor,indoor,steady periodic

[>

....

Exercise 7: *Find the phase delay (the difference, in hours), between the ambient temperature and the indoor steady periodic temperature. It is acceptable to approximate this from the display in Exercise 6 (clicking the mouse on a point of the display will yield the point coordinates in a small window at the top of your Maple program). It is even better if you can verify the phase delay by writing the steady-periodic solution in phase-amplitude form (see section 5.4 mechanical vibrations, page 324, for help).*

Solution: The Ambient temperature maximum is at hour 15. The corresponding indoor steady periodic maximum is (from a mouse click on the graph) about at hour 17.66. So the time delay is 2.66 hours, about 2 hours and 40 minutes.

You can check with phase amplitude formulas. We have
$$1.0785 \cos(0.26180\,t) + 10.782 \sin(0.26180\,t) = C \cos(0.26180\,t - \alpha)$$

⌐

```
> A:=1.0785;
  B:=10.782;
  C:=sqrt(A^2+B^2);
  alpha:=arctan(B/A); #phase
  delta:=alpha/.26180; #time delay (from midnight)
```

$$A := 1.0785$$

$$B := 10.782$$

$$C := 10.836$$

$$\alpha := 1.4711$$

$$\delta := 5.6192$$

```
> delta-3.;   #time delay!
```

$$2.6192$$

```
>
```

Exercise 8: *Suppose the inside temperature is 76 degrees when the furnace is turned off. Using 3-d plots and other commands (see hints at Professor Gustafson's page, or use Maple help), figure out for the range of insulation constants* $.2 < k < .48$ *, the times during the first 72 hours when the indoor temperature is at or below 30 degrees. (This could be bad for your water pipes.) Justify the logic and math used to find these "bad" times, in a short paragraph. Illustrate your work with a computer graphic.*

Solution:

```
> restart:
  Digits:=5:
> c[1]:=k/(k^2+omega^2)*(a[1]*k-b[1]*omega);
  d[1]:=k/(k^2+omega^2)*(a[1]*omega+b[1]*k);  #use book's letter
  c[0]:=u[0]-a[0]-c[1];
  U:=t->a[0]+c[0]*exp(-k*t)+c[1]*cos(omega*t)+d[1]*sin(omega*t);
```

$$c_1 := \frac{k\,(a_1\,k - b_1\,\omega)}{k^2 + \omega^2}$$

$$d_1 := \frac{k\,(a_1\,\omega + b_1\,k)}{k^2 + \omega^2}$$

$$c_0 := u_0 - a_0 - \frac{k\,(a_1\,k - b_1\,\omega)}{k^2 + \omega^2}$$

$$U := t \rightarrow a_0 + c_0\,\mathbf{e}^{(-k\,t)} + c_1\,\cos(\omega\,t) + d_1\,\sin(\omega\,t)$$

```
> omega:=evalf(Pi/12);
  a[0]:=35;
  a[1]:=-9.8995;
  b[1]:=-9.8995;
  u[0]:=76.;
```

$$\omega := 0.26180$$

$$a_0 := 35$$

$$a_1 := -9.8995$$

$$b_1 := -9.8995$$

$$u_0 := 76.$$

```
> U(t);
```

$$35 + \left(41. - \frac{k\,(-9.8995\,k + 2.5917)}{k^2 + 0.068539}\right)\mathbf{e}^{(-k\,t)} + \frac{k\,(-9.8995\,k + 2.5917)\cos(0.26180\,t)}{k^2 + 0.068539}$$

$$+ \frac{k\,(-2.5917 - 9.8995\,k)\sin(0.26180\,t)}{k^2 + 0.068539}$$

```
> UU:=(t,k)->35+(41.-k/(k^2+.68539e-1)*(-9.8995*k+2.5917))*exp(-k*t)
  +k/(k^2+.68539e-1)*(-9.8995*k+2.5917)*cos(.26180*t)+k/(k^2+.68539e
  -1)*(-2.5917-9.8995*k)*sin(.26180*t);   #pasted in from above
```

$$UU := (t, k) \rightarrow 35 + \left(41. - \frac{k\,(-9.8995\,k + 2.5917)}{k^2 + 0.068539}\right)\mathbf{e}^{(-k\,t)} + \frac{k\,(-9.8995\,k + 2.5917)\cos(0.26180\,t)}{k^2 + 0.068539}$$

$$+ \frac{k\,(-2.5917 - 9.8995\,k)\sin(0.26180\,t)}{k^2 + 0.068539}$$
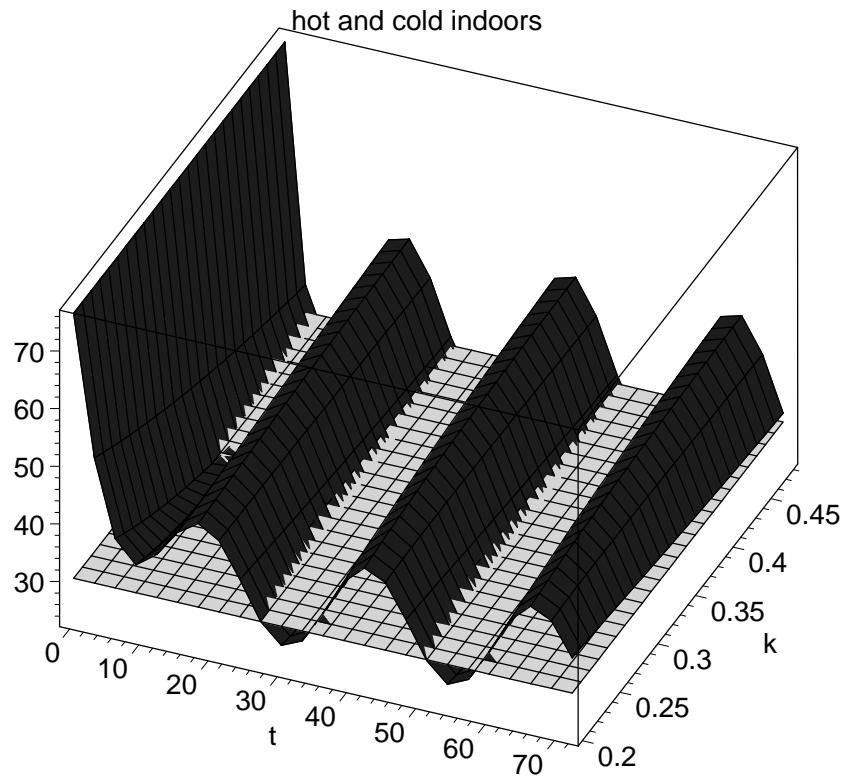
```
> with(plots):
> plot1:=plot3d(UU(t,k),t=0..72,k=.2..(.48),color=blue):
    #temperature plot, as it depends on time and insulation k
  plot2:=plot3d(30,t=0..72,k=.2..(.48),color=pink):
    #temperature of 30 degrees
  display({plot1,plot2},axes=boxed,title='hot and cold indoors');
```
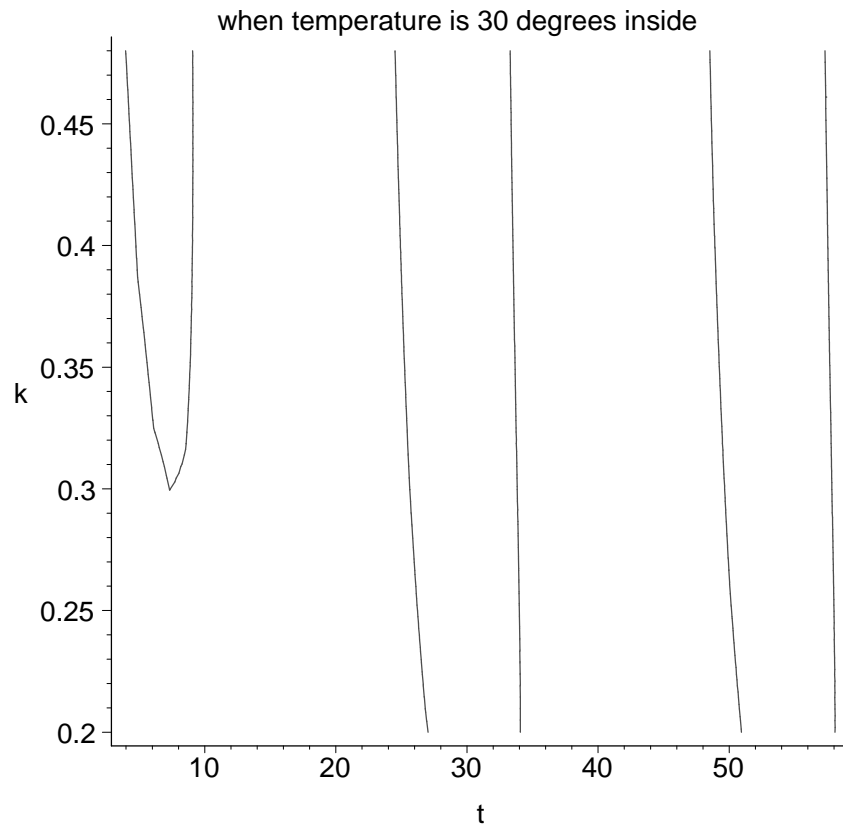
hot and cold indoors

```
> implicitplot(UU(t,k)=30,t=0..72,k=.2..(.48),grid=[60,60],
  title='when temperature is 30 degrees inside');
```

when temperature is 30 degrees inside

The 3-d plot gives an idea of when temperatures are under 30 degrees - it tends to be in the early morning hours. The second plot shows more precise details. The first night, when k<.3 the house stays above 30 degrees. The second and third nights all ranges of thermal conductivity k yield indoor temperatures under 30 degrees. For k=.2, the cold period is between about 3:00 a.m. and 10:00 a.m. on the second and third nights. For k=.48 the cold times are about 12:45 a.m thru about 9:20 a.m. For intermediate conductivities the cold times vary approximately linearly from the extremes. Notice that he more poorly insulated house gets colder sooner, but also warms up sooner in the morning.