

Name:

1. For each of the following systems of equations, find the augmented matrix corresponding to it. Then *either* reduce the matrix to the reduced row echelon form and use the result to solve the system, *or* reduce the matrix to the row echelon form and use backward substitution to solve the system.

(a)

$$2x_1 + 8x_2 + 3x_3 = 2$$

$$x_1 + 3x_2 + 2x_3 = 5$$

$$2x_1 + 7x_2 + 4x_3 = 8$$

Augmented Matrix:

$$\left[ \begin{array}{ccc|c} 2 & 8 & 3 & 2 \\ 1 & 3 & 2 & 5 \\ 2 & 7 & 4 & 8 \end{array} \right]$$

swap  
(R<sub>1</sub>, R<sub>2</sub>)

$$\left[ \begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 2 & 8 & 3 & 2 \\ 2 & 7 & 4 & 8 \end{array} \right]$$

-2R<sub>1</sub> + R<sub>2</sub>  
-2R<sub>1</sub> + R<sub>3</sub>

$$\left[ \begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & 2 & -1 & -8 \\ 0 & 1 & 0 & -2 \end{array} \right]$$

swap  
(R<sub>2</sub>, R<sub>3</sub>)

$$\left[ \begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 2 & -1 & -8 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & -1 & -4 \end{array} \right]$$

-2R<sub>2</sub> + R<sub>3</sub>

$$\left[ \begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{array} \right]$$

BY BACK-SUBSTITUTION, x<sub>3</sub> = 4, x<sub>2</sub> = -2, x<sub>1</sub> = -3 //

(b)

$$3x_1 - 6x_2 + x_3 + 13x_4 = 15$$

$$3x_1 - 6x_2 + 3x_3 + 21x_4 = 21$$

$$2x_1 - 4x_2 + 5x_3 + 26x_4 = 23$$

$$\left[ \begin{array}{cccc|c} 3 & -6 & 1 & 13 & 15 \\ 3 & -6 & 3 & 21 & 21 \\ 2 & -4 & 5 & 26 & 23 \end{array} \right]$$

$$R_1/3 \quad \left[ \begin{array}{cccc|c} 1 & -2 & 1/3 & 13/3 & 5 \end{array} \right]$$

$$R_2/3 \quad \left[ \begin{array}{cccc|c} 1 & -2 & 1 & 7 & 7 \end{array} \right]$$

$$R_3/2 \quad \left[ \begin{array}{cccc|c} 1 & -2 & 5/2 & 13 & 23/2 \end{array} \right]$$

$$\rightarrow \left[ \begin{array}{cccc|c} 1 & -2 & 1/3 & 13/3 & 5 \end{array} \right]$$

$$-R_1 + R_2 \quad \left[ \begin{array}{cccc|c} 0 & 0 & 2/3 & 8/3 & 2 \end{array} \right]$$

$$-R_1 + R_3 \quad \left[ \begin{array}{cccc|c} 0 & 0 & 13/6 & 29/3 & 13/2 \end{array} \right]$$

$$\begin{matrix} \rightarrow \\ \frac{3}{2}R_2 \\ \frac{5}{2}R_3 \end{matrix} \quad \left[ \begin{array}{cccc|c} 1 & -2 & 1/3 & 13/3 & 5 \\ 0 & 0 & 1 & 4 & 3 \\ 0 & 0 & 1 & 4 & 3 \end{array} \right]$$

$$-R_2 + R_3 \quad \left[ \begin{array}{cccc|c} 1 & -2 & 1/3 & 13/3 & 5 \\ 0 & 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$\frac{1}{3}R_2 + R_1 \quad \left[ \begin{array}{cccc|c} 1 & -2 & 0 & 3 & 4 \\ 0 & 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Free variables: x<sub>2</sub>, x<sub>4</sub>

"s" "t"

$$x_1 = 4 - 3t + 2s$$

$$x_3 = 3 - 4t //$$

2. Use Gauss-Jordan elimination to find the inverse of the following matrix:

$$\begin{array}{l}
 \left( \begin{array}{ccc} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 2 & 3 & 1 \end{array} \right) \\
 \left[ \begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & 1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{-2R_1 + R_3} \left[ \begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 & 1 \end{array} \right] \xrightarrow{\text{swap}} \left[ \begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 0 & 1 \\ 0 & 2 & 1 & 0 & 1 & 0 \end{array} \right] \xrightarrow{(R_2, R_3)} \left[ \begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 0 & 1 \\ 0 & 2 & 1 & 0 & 1 & 0 \end{array} \right] \\
 \xrightarrow{-2R_2 + R_3} \left[ \begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 0 & 1 \\ 0 & 0 & -1 & 4 & 1 & -2 \end{array} \right] \xrightarrow{-R_2} \left[ \begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 0 & 1 \\ 0 & 0 & -1 & 4 & 1 & -2 \end{array} \right] \xrightarrow{-R_3 + R_2} \left[ \begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & -1 & 4 & 1 & -2 \end{array} \right] \xrightarrow{R_2 + R_1} \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -2 & 1 & -1 \\ 0 & 0 & -1 & 4 & 1 & -2 \end{array} \right] \\
 \text{The inverse is: } \left[ \begin{array}{ccc} -1 & -1 & 1 \\ 2 & 1 & -1 \\ -4 & -1 & 2 \end{array} \right] \quad // 
 \end{array}$$

3. Compute the determinants of each of the following matrices and use it to determine whether the matrix is singular or nonsingular (know what nonsingularity implies about a matrix!!):

(a)

$$\left( \begin{array}{cc} 2 & -3 \\ 2 & 3 \end{array} \right)$$

$$\det \begin{bmatrix} 2 & -3 \\ 2 & 3 \end{bmatrix} = 2 \cdot 3 - 2(-3) = 6 + 6 = 12$$

The det is nonzero.  $\Rightarrow$  Nonsingular.

$\Rightarrow$  Matrix is invertible

$\Rightarrow$  unique soln to  $Ax = b$  exists...

(b)

$$\left( \begin{array}{cccc} 1 & 0 & 1 & 0 \\ 1 & 2 & 2 & 2 \\ 0 & 3 & 1 & 4 \\ 1 & 0 & 4 & 0 \end{array} \right)$$

Here, we could use row operations to simplify...

$(-R_1 + R_2 \text{ and } -R_1 + R_4)$

$$\Rightarrow \left[ \begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 0 & 3 & 0 \end{array} \right]$$

Now expanding about column 1,

$$(-1)^{1+1} \cdot \det \left| \begin{array}{ccc} 2 & 1 & 2 \\ 3 & 1 & 4 \\ 0 & 3 & 0 \end{array} \right|$$

$$\text{expand about } R_3 \Rightarrow -1 \cdot 3 \cdot \det \left| \begin{array}{cc} 2 & 2 \\ 3 & 4 \end{array} \right|$$

$$= -3(8 - 6) = -6$$

$$\therefore (1)(-1)^2 \cdot (-6) = -6 \quad //$$

4. Consider the following linear system of equations

$$\begin{aligned} 3x_1 + x_2 + 2x_3 &= 5 \\ x_1 + x_2 + x_3 &= 2 \\ x_1 + 2x_2 + 2x_3 &= 1 \end{aligned}$$

(a) Write down the corresponding matrix equation for the system above

$$\left[ \begin{array}{ccc|c} 3 & 1 & 2 & 5 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 2 & 1 \end{array} \right] \left[ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right] = \left[ \begin{array}{c} 5 \\ 2 \\ 1 \end{array} \right]$$

$\leftarrow \text{This is the } \underline{\text{matrix equation}}.$   
(This is what the question is asking you to do.)

NOTE: The "augmented form" is  $\left[ \begin{array}{ccc|c} 3 & 1 & 2 & 5 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 2 & 1 \end{array} \right]$   $\leftarrow$  (This would not receive full credit on a test asking you to do this...)

(b) Compute the determinant of the  $3 \times 3$  coefficient matrix. What does the value of the determinant tell you about the solution of the system? In particular, can you rule out if the system has one solution, no solution, or infinitely many solutions?

$$\left[ \begin{array}{ccc|c} 3 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \end{array} \right] \quad 6 + 1 + 4 - 2 - 6 - 2 = 1$$

Notice that the question didn't ask you to find the determinant via cofactor expansion, so the "quick method" is O.K. to use.

( $\Rightarrow$  I might specifically ask for cofactor expansion method, in which case you would need to work out the det. as in problem 3b, but unless you are explicitly asked, it's ok.)

Since the  $\det \neq 0$ , the inverse exists,

and solution is unique.

(Make sure you understand this concept, as well as properties of Nonsingular matrices we discussed in class...)

(c) Find the inverse of the coefficient matrix.

I choose the  $\frac{1}{\det(A)} [A^{-1}]^T$  method. Gauss-Jordan on  $[A : I]$  also works ..

$$\frac{1}{\det(A)} \begin{bmatrix} 3 \cdot (-1)^{1+1} & \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} & 1 \cdot (-1)^{1+2} & \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} & 2 \cdot (-1)^{1+3} & \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \\ -1 \cdot (2-1) = -1 & & & & 2 \cdot (2-1) = & \\ = 3 \cdot 0 = 0 & & & & & \\ 1 \cdot (-1)^{2+1} & \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} & 1 \cdot (-1)^{2+2} & \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} & 1 \cdot (-1)^{2+3} & \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} \\ -1 \cdot (2-4) = 2 & & +1 \cdot (6-2) = 4 & & -1 \cdot (6-1) = -5 & = \frac{1}{1} \begin{bmatrix} 0 & 2 & -1 \\ -1 & 4 & -1 \\ 1 & -5 & 2 \end{bmatrix} \\ 1 \cdot (-1)^{3+1} & \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} & 2 \cdot (-1)^{3+2} & \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} & 2 \cdot (-1)^{3+3} & \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} \\ = 1 \cdot (1-2) = -1 & & 2 \cdot -1 \cdot (3-2) = & & 2 \cdot (3-1) & \\ = -1 & & & & & \end{bmatrix}$$

INVERSE IS :  $\begin{bmatrix} 0 & 2 & -1 \\ -1 & 4 & -1 \\ 1 & -5 & 2 \end{bmatrix}$

(d) Find the solution to the linear system.

To solve  $Ax = b$ , use  $A^{-1}$

$$\hat{x} = A^{-1} b$$

$$= \begin{bmatrix} 0 & 2 & -1 \\ -1 & 4 & -1 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}$$

$\uparrow$   
unique solution  
to our system.

5. Let A be the matrix:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(a) Show that this matrix is row equivalent to the  $2 \times 2$  identity matrix provided that  $ad - bc \neq 0$ .

$$\text{If } a \neq 0, \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{R_1/a} \begin{bmatrix} 1 & b/a \\ c & d \end{bmatrix} \xrightarrow{cR_1 + R_2} \begin{bmatrix} 1 & b/a \\ 0 & d - cb/a \end{bmatrix}$$

$$\xrightarrow{\quad} \begin{bmatrix} 1 & b/a \\ 0 & \frac{ad - cb}{a} \end{bmatrix} \xrightarrow{\quad} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Now if  $a = 0$ , then  $c \neq 0$  if  $ad - bc \neq 0$ . Row swap  $\begin{bmatrix} c & d \\ a & b \end{bmatrix}$

$$\frac{1}{c}R_1 \begin{bmatrix} 1 & d/c \\ a & b \end{bmatrix} \xrightarrow{-aR_1 + R_2} \begin{bmatrix} 1 & d/c \\ 0 & b - ad/c \end{bmatrix} \xrightarrow{\quad} \begin{bmatrix} 1 & 0 \\ 0 & \frac{bc - ad}{c} \end{bmatrix} \dots \xrightarrow{\quad} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (\text{see HW # 32 in §3.3})$$

(b) What is the inverse of A?

Augmented matrix:  $\begin{bmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{bmatrix} \xrightarrow{R_2 - \frac{c}{a}R_1} \begin{bmatrix} a & b & 1 & 0 \\ 0 & \frac{ad - bc}{a} & \frac{c}{a} & 1 \end{bmatrix} \xrightarrow{R_1/a} \begin{bmatrix} 1 & b/a & 1/a & 0 \end{bmatrix}$

$$\xrightarrow{-\frac{b}{a}R_2 + R_1} \begin{bmatrix} 1 & 0 & \frac{d}{ad - bc} & \frac{-b}{ad - bc} \\ 0 & 1 & \frac{-c}{ad - bc} & \frac{a}{ad - bc} \end{bmatrix} \Rightarrow A^{-1} \text{ is } \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

(from notes in section 3.5)

6. Using cofactor expansion, find the determinant of the matrix

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 3 & 0 & 4 \\ 5 & 0 & 6 & 0 \\ 0 & 7 & 0 & 8 \end{pmatrix}$$

Expanding about the 1st row,

$$1(-1)^{1+1} \det \underbrace{\begin{vmatrix} 3 & 0 & 4 \\ 0 & 6 & 0 \\ 7 & 0 & 8 \end{vmatrix}}_{6(-1)^{2+2} \det \begin{vmatrix} 3 & 4 \\ 7 & 8 \end{vmatrix}} + 0 + 2(-1)^{1+3} \det \underbrace{\begin{vmatrix} 0 & 3 & 4 \\ 5 & 0 & 0 \\ 0 & 7 & 8 \end{vmatrix}}_{5(-1)^{2+1} \det \begin{vmatrix} 3 & 4 \\ 7 & 8 \end{vmatrix}} + 0$$

$$6(24 - 28) \quad 5(-1)(24 - 28)$$

$$-24 \quad 20$$

$$1(-24) + 2(20) = \underline{16}$$

7. Solve the linear system  $Ax = b$  using Cramer's rule, where A is:

$$\begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & 2 \\ -1 & -2 & 1 \end{pmatrix}$$

Note that you can  
find determinants here  
however you like!

and b is:

$$\begin{pmatrix} -7 \\ 0 \\ 12 \end{pmatrix}$$

$$\det(A) = 1(-1)^{1+1} \begin{vmatrix} 0 & 2 \\ -2 & 1 \end{vmatrix} + 2(-1)^{1+2} \begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix} = 1 \cdot 4 - 2 \cdot 5 = -6$$

$$x_1 = \frac{\det \begin{vmatrix} -7 & 2 & 0 \\ 0 & 0 & 2 \\ 12 & -2 & 1 \end{vmatrix}}{\det(A)} = \frac{2(-1)^{2+3} \begin{vmatrix} -7 & 2 \\ 12 & 2 \end{vmatrix}}{-6} = \frac{-2(14 - 24)}{-6} = -\frac{10}{3}$$

$$x_2 = \frac{\det \begin{vmatrix} 1 & -7 & 0 \\ 3 & 0 & 2 \\ -1 & 12 & 1 \end{vmatrix}}{\det(A)} = \frac{1(-1)^{1+1} \begin{vmatrix} 0 & 2 \\ 12 & 1 \end{vmatrix} - 7(-1)^{1+2} \begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix}}{-6} = \frac{1(-24) + 7(5)}{-6} = -\frac{11}{6}$$

$$x_3 = \frac{\det \begin{vmatrix} 1 & 2 & -7 \\ 3 & 0 & 0 \\ -1 & -2 & 12 \end{vmatrix}}{\det(A)} = \frac{3(-1)^{2+1} \begin{vmatrix} 2 & -7 \\ -2 & 12 \end{vmatrix}}{-6} = \frac{-3(24 - 14)}{-6} = 5.$$

8. Find the solution subspace of the linear system  $Ax = 0$ , where

$$A =$$

$$\begin{pmatrix} 2 & 4 & -2 & 0 & -16 \\ 4 & 0 & 4 & -8 & 0 \end{pmatrix} \xrightarrow{\text{row reduce...}} \begin{bmatrix} 1 & 2 & -1 & 0 & -8 \\ 0 & 1 & -1 & 1 & -4 \end{bmatrix}$$

$x_1$  and  $x_2$  are the fixed variables and  $x_3, x_4$ , and  $x_5$  are free.

$$R_2 \text{ implies } x_2 = x_3 - x_4 + 4x_5$$

$$\begin{aligned} \text{and } R_1 \text{ implies } x_1 &= -2x_2 + x_3 + 8x_5 = -2(x_3 - x_4 + 4x_5) + x_3 + 8x_5 \\ &= -x_3 + 2x_4 \end{aligned}$$

$$\text{SOLN LOOKS LIKE: } \underline{x} = x_3 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 4 \\ 0 \\ 0 \end{bmatrix} \quad \begin{array}{l} \text{so the soln.} \\ \text{subspace is} \\ \text{spanned by...} \end{array} \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 0 \\ 0 \end{bmatrix} \right\}$$

9. Consider the matrix below.

$$A = \begin{pmatrix} 3 & -1 & 7 & 3 & 9 \\ -2 & 2 & -2 & 7 & 5 \\ -5 & 9 & 3 & 3 & 4 \\ -2 & 6 & 6 & 3 & 7 \end{pmatrix} \quad (1)$$

with reduced row echelon form,

$$B = rref(A) = \begin{pmatrix} 1 & 0 & 3 & 0 & 5 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (2)$$

Find the basis for the solution space of the given homogeneous linear system. What is the dimension of the nullspace? What is the rank of the coefficient matrix?

Rank is 3.  $n = 5$ , so the dimension of  $\text{Null}(A) = 2$ .

Since  $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$ , the free variables are  $x_3$  and  $x_5$ .  
Let  $x_3 = s$  and  $x_5 = t$ .

$$x_1 = -3s - \frac{5}{2}t$$

$$x_2 = -2s - \frac{3}{2}t$$

$$x_3 = s$$

$$x_4 = -t$$

$$x_5 = t$$

Every soln  
is of the  
form  $\rightarrow$

$$\underline{x} = s \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5/2 \\ -3/2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

(Every vector of the solution space of  $A\underline{x} = 0$  can be written in this form.)

The dimension of the  $\text{null}(A) = 2$ . //