

Math 2250-3

HW due Monday Nov 3

6.5.6 (3, 5, 7, 14, 16, 21, 22)

$$3. \begin{cases} x'' + 100x = 15 \cos 5t + 20 \sin 5t \\ x(0) = 25 \\ x'(0) = 0 \end{cases}$$

$$x_H: c_1 \cos 10t + c_2 \sin 10t \quad (\omega_0 = \sqrt{\frac{k}{m}} = 10)$$

$$100(x_p = A \cos 5t + B \sin 5t)$$

$$+ 0(x_p' = -5A \sin 5t + 5B \cos 5t)$$

$$+ 1(x_p'' = -25A \cos 5t - 25B \sin 5t)$$

$$Lx_p = \cos 5t (100A - 25A) = \cos 5t (15) \quad 75A = 15; A = 1/5$$

$$+ \sin 5t (100B - 25B) = \sin 5t (20) \quad 75B = 20; B = 4/15$$

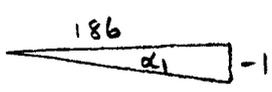
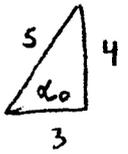
$$x(t) = \frac{1}{5} \cos 5t + \frac{4}{15} \sin 5t + c_1 \cos 10t + c_2 \sin 10t$$

$$x(0) = 25 = \frac{1}{5} + c_1 \quad c_1 = 24 \frac{4}{5} = \frac{124}{5}$$

$$x'(0) = 0 = \frac{20}{15} + 10c_2 \quad c_2 = -\frac{2}{15}$$

$$x(t) = \frac{1}{5} \cos 5t + \frac{4}{15} \sin 5t + \frac{124}{5} \cos 10t - \frac{2}{15} \sin 10t$$

$$= \frac{1}{15} (3 \cos 5t + 4 \sin 5t) + \frac{2}{15} (186 \cos 10t - \sin 10t)$$



$$x(t) = \frac{1}{3} \cos(5t - \alpha_0) + \frac{2}{15} \sqrt{(186)^2 + 1} \cos(10t - \alpha_1)$$

$$\alpha_0 = \tan^{-1}\left(\frac{4}{3}\right) \approx 0.927 \text{ rad}$$

$$\alpha_1 = \tan^{-1}\left(-\frac{1}{186}\right) \approx -0.0054 \text{ rad}$$

$$5. \begin{cases} mx'' + kx = F_0 \cos \omega t \quad \omega \neq \omega_0; \omega_0 = \sqrt{k/m} \\ x(0) = x_0 \\ x'(0) = 0 \end{cases}$$

We (and book p.340), found $x_p = \frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t$,

$$\text{so } x = x_p + x_H$$

$$x = \frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t + c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$$

$$x(0) = x_0 = \frac{F_0/m}{\omega_0^2 - \omega^2} + c_1; \quad c_1 = x_0 + \frac{F_0/m}{\omega_0^2 - \omega^2}$$

$$x'(0) = 0 = \omega_0 c_2; \quad c_2 = 0$$

$$\text{so } x(t) = \frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t + \left(x_0 + \frac{F_0/m}{\omega_0^2 - \omega^2}\right) \cos \omega_0 t$$

7. $x'' + 4x' + 4x = 10 \cos 3t$
 $x_{sp} = A \cos 3t + B \sin 3t$;

we could use the results of derivation p.346,
 but it is good practice to work it out:

4($x_p = A \cos 3t + B \sin 3t$
 + 4($x_p' = -3A \sin 3t + 3B \cos 3t$
 + 1($x_p'' = -9A \cos 3t - 9B \sin 3t$

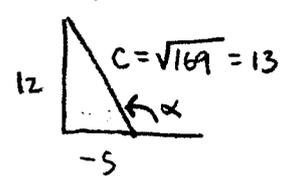
$L(x_p) = \cos 3t [4A + 12B - 9A] = \cos 3t [10]$
 $+ \sin 3t [4B - 12A - 9B] + \sin 3t [0]$

$-5A + 12B = 10$
 $-12A - 5B = 0$

$\begin{bmatrix} -5 & 12 \\ -12 & -5 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$

$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{25+144} \begin{bmatrix} -5 & -12 \\ 12 & -5 \end{bmatrix} \begin{bmatrix} 10 \\ 0 \end{bmatrix}$
 $= \begin{bmatrix} -50 \\ 120 \\ 120 \\ 169 \end{bmatrix} = \frac{10}{169} \begin{bmatrix} -5 \\ 12 \\ 12 \end{bmatrix}$

$x_{sp}(t) = \frac{10}{169} (-5 \cos 3t + 12 \sin 3t)$



$= \frac{10}{169} 13 \cos(3t - \alpha) \quad \alpha = \cos^{-1} \left(\frac{-5}{13} \right)$

$x_{sp}(t) = \frac{10}{13} \cos(3t - \alpha), \quad \alpha = \cos^{-1} \left(\frac{-5}{13} \right)$
 $\alpha \approx 1.966 \text{ rad}$

14. $x'' + 8x' + 25x = 5 \cos t + 13 \sin t$
 $x(0) = 5$
 $x'(0) = 0$

25($x_p = A \cos t + B \sin t$
 8($x_p' = -A \sin t + B \cos t$
 1($x_p'' = -A \cos t - B \sin t$

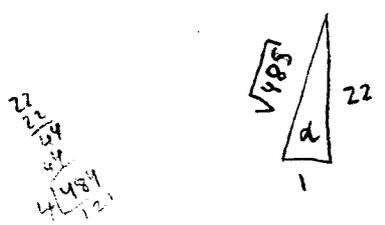
$24A + 8B = 5$
 $-8A + 24B = 13$

$\begin{bmatrix} 24 & 8 \\ -8 & 24 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \end{bmatrix}$

$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{4 \cdot 144 + 64} \begin{bmatrix} 24 & -8 \\ 8 & 24 \end{bmatrix} \begin{bmatrix} 5 \\ 13 \end{bmatrix}$

$L(x_p) = \cos t [25A + 8B - A] = \cos t [5]$
 $+ \sin t [25B - 8A - B] + \sin t [13]$

$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{640} \begin{bmatrix} 16 \\ 352 \end{bmatrix} = \frac{1}{40} \begin{bmatrix} 1 \\ 22 \end{bmatrix}$



$x_{sp}(t) = \frac{1}{40} (\cos t + 22 \sin t)$

$x_{sp} = \frac{\sqrt{485}}{40} \cos(t - \alpha); \quad \alpha = \tan^{-1}(22) \approx 1.525 \text{ rad}$

cont'd.

14 cont'd.

$$x_H: e^{rt} : p(r) = r^2 + 8r + 25 = 0$$

$$= (r+4)^2 + 9$$

$$= (r+4+3i)(r+4-3i)$$

$$r = -4 \pm 3i$$

$$x_H(t) = e^{-4t} (c_1 \cos 3t + c_2 \sin 3t)$$

you could get this far very easily in Maple:

> deqn := diff(x(t), t, t) + 8 * diff(x(t), t) + 25 * x(t) = 5 * cos(t) + 13 * sin(t)

> ics := x(0) = 5, D(x)(0) = 0;

> dsolve({deqn, ics}, x(t), method = laplace)

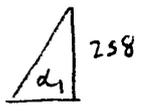
$$x(t) = \frac{1}{40} (\cos t + 22 \sin t) + e^{-4t} (c_1 \cos 3t + c_2 \sin 3t)$$

$$x(0) = 5 = \frac{1}{40} + c_1 ; \quad c_1 = 4 \frac{39}{40} = \frac{199}{40}$$

$$x'(0) = 0 = \frac{22}{40} - 4c_1 + 3c_2 ; \quad 3c_2 = 4c_1 - \frac{22}{40} = \frac{796-22}{40} = \frac{774}{40} ; \quad c_2 = \frac{258}{40}$$

so

$$x(t) = \frac{\sqrt{485}}{40} \cos(t - \alpha_0) + \frac{1}{40} e^{-4t} [199 \cos 3t + 258 \sin 3t]$$



$$= \frac{\sqrt{485}}{40} \cos(t - \alpha_0) + \frac{\sqrt{106165}}{40} e^{-4t} \cos(3t - \alpha_1) \quad \alpha_1 = \arctan\left(\frac{258}{199}\right)$$

$$x(t) \approx 0.5506 \cos(t - 1.525) + 8.1457 e^{-4t} \cos(3t - 0.9138)$$

16. $m x'' + c x' + k x = 10 \cos \omega t$

$m=1, c=4, k=5, F_0=10$

eqn (21) p. 346:

$$C(\omega) = \frac{F_0}{\sqrt{(k-m\omega^2)^2 + (c\omega)^2}} = \frac{10}{\sqrt{(5-\omega^2)^2 + 16\omega^2}}$$

practical resonance looks unlikely, since denom doesn't get very small.

any local max? $C'(\omega) = 0$

$$= (10)(-\frac{1}{2}) \left((5-\omega^2)^2 + 16\omega^2 \right)^{-3/2} \left(2(5-\omega^2)(-2\omega) + 32\omega \right)$$

$\neq 0 \quad \neq 0 \quad = 0?$

$$\omega [-4(5-\omega^2) + 32]$$
$$\omega [12 + 4\omega^2] \text{ nope}$$

shows $C'(\omega) < 0$ for $\omega > 0$
no practical resonance.

Let $x =$ horiz. displacement (to right)

21. $E = \frac{1}{2} k x^2 + \frac{1}{2} m v^2 + mgh$

$x = L \sin \theta$
 $v = L \theta'$



$h = L(1 - \cos \theta)$

$$E = \frac{1}{2} k L^2 \sin^2 \theta + \frac{1}{2} m (L \theta')^2 + mgL(1 - \cos \theta)$$

$$0 \equiv E'(t) = k L^2 \sin \theta \cos \theta + m L^2 \theta' \theta'' + mgL \sin \theta \theta'$$
$$= L \theta' [k L \sin \theta \cos \theta + m L \theta'' + mg \sin \theta]$$

\approx (linearize)

$$L \theta' [k L \theta + m L \theta'' + mg \theta]$$

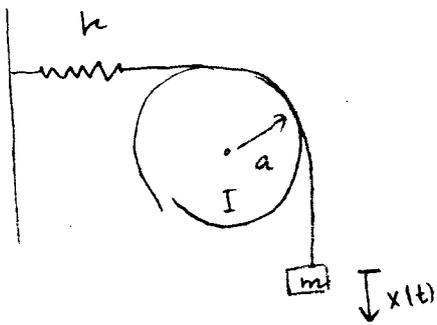
deduce $m L \theta'' + (mg + kL) \theta = 0$

$$\theta'' + \left(\frac{mg + kL}{mL} \right) \theta = 0$$

$$\theta'' + \left(\frac{g}{L} + \frac{k}{m} \right) \theta = 0$$

$$\omega_0 = \sqrt{\frac{g}{L} + \frac{k}{m}}$$

22.



$$E = \frac{1}{2} k x^2 + mgh + \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2$$

$$x' = v = a\omega$$

$$h = -x$$

$$E = \frac{1}{2} k x^2 - mgx + \frac{1}{2} m (x')^2 + \frac{1}{2} I \left(\frac{x'}{a}\right)^2$$

$$0 \equiv E' = kxx' - mgx' + mx'x'' + I/a^2 x'x'' \\ = x' [kx - mg + mx'' + I/a^2 x'']$$

deduce $(m + I/a^2)x'' + kx = mg$

so, for x_H , $\omega_0^2 = \frac{k}{m + I/a^2}$

(5)