

Determining whether or not for given vectors in  $\mathbb{R}^n$  form a basis for  $\mathbb{R}^n$ .

1]  $v_1 = (4, 7), v_2 = (5, 6)$

$\begin{bmatrix} 4 & 5 \\ 7 & 6 \end{bmatrix} \stackrel{+}{=} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  Since these are not scalar multiples, they are L.I. Another way,  
 $= 24 - 35 \neq 0 \Rightarrow$  L.I. By theorem 3 (p. 262),

Since  $S = \{v_1, v_2\}$  is L.I. and consists of 2 vectors,  
these vectors form a basis for  $\mathbb{R}^2$ . //

3]  $v_1 = (1, 7, -3), v_2 = (2, 1, 4), v_3 = (6, 5, 1), v_4 = (0, 7, 13)$

Any collection of 4 vectors in  $\mathbb{R}^3$  must be dependent.

Therefore, these vectors cannot form a basis for  $\mathbb{R}^3$ . //

8]  $v_1 = (2, 0, 0, 0), v_2 = (0, 3, 0, 0), v_3 = (0, 0, 7, 6), v_4 = (0, 0, 4, 5)$

If  $A = [v_1 | v_2 | v_3 | v_4]$  is nonsingular, the vectors are L.I.

Clearly  $A$  can be reduced to the identity matrix.

$A$  is Nonsingular.  $\Rightarrow (v_1, v_2, v_3, v_4)$  form a basis for  $\mathbb{R}^4$ . //

15]  $x_1 - 2x_2 + 3x_3 = 0$  (We are finding a basis for  
 $2x_1 - 3x_2 - x_3 = 0$  the solution subspace)

The coefficient matrix:  $\begin{bmatrix} 1 & -2 & 3 \\ 2 & -3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -11 \\ 0 & 1 & -7 \end{bmatrix} \Rightarrow x = t \begin{bmatrix} 11 \\ 7 \\ 1 \end{bmatrix}$   
so the solution subspace is spanned by //.

21]  $A = \begin{bmatrix} 1 & -4 & -3 & -7 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & 3 & 11 \end{bmatrix}$  rref(A)  $\rightarrow \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Backsolving, we get:  $x_4 = t$   
 $x_3 = s$   
 $x_2 = -s - 3t$   
 $x_1 = -s - 5t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

So the basis of our solution subspace:  $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

$$\underline{1} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & -9 \\ 2 & 8 & 2 \end{bmatrix} \quad \text{rref}(A) \Rightarrow \begin{bmatrix} 1 & 0 & 11 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

A nice basis of rowspace =  $\{[1, 0, 11], [0, 1, -4]\}$

Also, from the rref(A), we can see  $\begin{bmatrix} 3 \\ -9 \\ 2 \end{bmatrix} = 11\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - 4\begin{bmatrix} 2 \\ 5 \\ 5 \end{bmatrix}$

We can therefore toss this dependent<sup>†</sup> vector,

and the basis for our column space is:  $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 5 \end{bmatrix} \right\}$

$$\underline{15} \quad A = \begin{bmatrix} 3 & 2 & 4 & 1 \\ 2 & 1 & 3 & 2 \\ 2 & 2 & 2 & 3 \\ 2 & 1 & 3 & 4 \end{bmatrix} \quad \text{rref}(A) \Rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = E$$

from rref(A), we can deduce  $v_3 = 2v_1 - v_2$  (toss this out!)

So  $v_1, v_2$ , and  $v_4$  are linearly independent., and therefore  $\{v_1, v_2, v_4\}$  form a basis for  $\mathbb{R}^3$ .

Hence  $\text{Col}(E)$  is a 3-dimensional subspace of  $\mathbb{R}^4$  and the column rank of  $E$  is 3. (see Ex 3, page 259) //

$$\underline{16} \quad A = \begin{bmatrix} 5 & 3 & 7 & 1 & 5 \\ 4 & 1 & 7 & -1 & 4 \\ 2 & 2 & 2 & 2 & 6 \\ 2 & 3 & 1 & 4 & 7 \end{bmatrix} \quad \text{rref}(A) \Rightarrow E = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

From  $E$ , we can again see that  $v_3 = 2v_1 - v_2$ , so  $v_3$  is dependent on these two. Get rid of it.

Now  $v_1, v_2, v_4$ , and  $v_5$  are independent,

Therefore,  $\{v_1, v_2, v_4, v_5\}$  are a basis for  $\mathbb{R}^4$  (which is their span, since 4 independent vectors in  $\mathbb{R}^4$  span it).

26] Explain why the  $n \times n$  matrix  $A$  is invertible if and only if its rank is  $n$ .

Soln:  $A$  is invertible  $\Leftrightarrow$  rowspace is  $n$ -dimensional  
 $\text{Rank}(A) = n$ , since the rowspace is the span of rows of  $\text{rref}(A)$ . //

27] Let  $A$  be a  $3 \times 5$  matrix whose 3 row vectors are L.I. Prove that for each  $b$  in  $\mathbb{R}^3$ , the nonhomogeneous system  $Ax = b$  has a solution.

Soln: If  $A$ 's 3 rows are L.I.,  
 $\Rightarrow$  rowspace is 3-D  
 $\Rightarrow \text{rref}(A)$  has no zero rows,  
 $\Rightarrow \text{rref}(A|b)$  is consistent

Therefore, any given vector  $b$  in  $\mathbb{R}^3$  can be expressed as a linear combination  $b = x_1\mathbf{q}_1 + x_2\mathbf{q}_2 + \dots + x_5\mathbf{q}_5$  of the column vectors of  $A$ .

The column vector  $x$  whose elements are the coefficients in this linear combination is then a solution of the equation  $Ax = b$ . //

28] Let  $A$  be a  $5 \times 3$  matrix that has 3 L.I. row vectors. Suppose that  $b$  is a vector in  $\mathbb{R}^5$  such that the nonhomogeneous system  $Ax = b$  has a solution. Prove that the solution is unique.

Soln: 3 L.I. rows  
 $\Rightarrow$  rowspace is 3D.

$$\Rightarrow \text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{Thus, } \text{rref}(A|b) = \begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \end{bmatrix}$$

} The system is consistent,  
} meaning that these last 2 entries are 0, and the first 3 rows determine

$x_1, x_2, x_3$  uniquely.

Alternatively,

$\text{Rank}(A) = 3 \Rightarrow$  column vectors  $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$  are L.I. Therefore, any given vector  $b$  in  $\mathbb{R}^5$  can be expressed in at most one way as a lin. comb.  $b = x_1\mathbf{q}_1 + x_2\mathbf{q}_2 + x_3\mathbf{q}_3$  of the col. vctrs of  $A$   $\Rightarrow Ax = b$  has at most one solution.

5] The set of all  $f$  such that  $f(0) = 0$ .

YES. a) If  $f(0) = 0$  and  $g(0) = 0$ , then  
 $(f+g)(0) = f(0) + g(0) = 0$

so  $\vee$  is closed under addition.

b) if  $f(0) = 0$  then  $(cf)(0) = c \cdot f(0) = c \cdot 0 = 0$ .  
 so  $\vee$  is closed under scalar multiplication.

7] The set of all  $f$  such that  $f(0) = 0$  and  $f(1) = 1$ .

NO. Does not contain the zero function.

Also, if  $f(1) = 1$  and  $g(1) = 1$ ,  $(f+g)(1) = 2$ .  
 Not closed under addition.

8] The set of all  $f$  such that  $f(-x) = -f(x) \forall x$ .

YES.

a) if  $f(-x) = -f(x)$  and  $g(-x) = -g(x)$ ,  
 $(f+g)(-x) = f(-x) + g(-x) = -f(x) - g(x)$   
 $= -(f(x) + g(x)) = -(f+g)(x) \checkmark$

b) if  $f(-x) = -f(x)$ ,  
 $(cf)(-x) = c f(-x) = c (-f(x)) = -c f(x) = -(cf)(x) \checkmark$

13] Determine if  $\sin x$  and  $\cos x$  are L.I.

If  $c_1 \sin x + c_2 \cos x = 0$  then  $c_1 \sin x = -c_2 \cos x$ ,  
 so  $c_1 \tan x = -c_2$ .

At  $x = 0 \Rightarrow c_2 = 0$ .  $\therefore$  Independent.

At  $x = \pi/4 \Rightarrow c_1 = 0$ .

(Also,  $\sin x$  &  $\cos x$  are most certainly NOT scalar multiples of each other, so not dependent.)

14]  $e^x$  and  $x e^x$  are L.I., because they are not scalar multiples of each other.

15]  $1+x, 1-x, 1-x^2$

Let  $c_1(1+x) + c_2(1-x) + c_3(1-x^2) = 0$

Collecting like terms,

$$(c_1 + c_2 + c_3)1 + (c_1 - c_2)x - c_3 x^2 = 0$$

Since  $\{1, x, x^2\}$  are L.I.,  $c_1 + c_2 + c_3 = 0$

$$c_1 - c_2 = 0$$

$$-c_3 = 0$$

$$\begin{vmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{vmatrix} \text{ rref } \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix} \Rightarrow c_1 = c_2 = c_3 = 0$$

$\therefore$  Independent.

$$19) \frac{x-5}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3} = \frac{A(x-3) + B(x-2)}{(x-2)(x-3)}$$

$x-5 = (A+B)x + (-3A - 2B)$

Linear Independence of  $\{1, x, x^2\}$  requires the respective coefficients are equal:

$$\begin{matrix} x: & A+B = 1 \\ 1: & -3A - 2B = -5 \end{matrix} \quad \begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix} \Rightarrow \begin{matrix} A = 3 \\ B = -2 \end{matrix}$$

$$21) \frac{8}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4} = \frac{A(x^2+4) + (Bx+C)x}{x(x^2+4)}$$

$$\begin{matrix} x: & 8 = A(x^2+4) + (Bx+C)x \\ x^2: & 0 = A + B \\ x: & 0 = C \\ 1: & 8 = 4A \end{matrix} \Rightarrow \begin{matrix} B = -2 \\ C = 0 \\ A = 2 \end{matrix}$$

23)  $y''' = 0$ , integrate 3 times

$$y'' = c_1$$

$$y' = c_1 x + c_2$$

$$y = \frac{1}{2}c_1 x^2 + c_2 x + c_3$$

$$= ax^2 + bx + c$$

Solution space =  $\text{span } \{1, x, x^2\}$   
basis.

$$25) y'' - 5y' = 0$$

$$\text{let } v = y'$$

$$\text{Then } v' - 5v = 0$$

$$v' = 5v$$

$$v(x) = ce^{5x}$$

$$y'(x) = ce^{5x}$$

$$\text{Integrate: } y(x) = \frac{c}{5}e^{5x} + d = ae^{5x} + b$$

$$\text{Solution space} = \text{span } \{1, e^{5x}\}$$

(basis)