

Problems 11 - 20, determine whether Theorem 1 does or does not guarantee existence of a solution of the given initial value problem. If existence is guaranteed, determine whether Theorem 1 does or does not guarantee uniqueness of that solution.

11)  $\frac{dy}{dx} = 2x^2y^3 ; y(1) = -1$

Soln:  $f(x,y)$  and  $\frac{\partial f}{\partial y} = 6x^2y^2$  are both continuous everywhere in the  $xy$ -plane, so the IVP has a unique solution.

12)  $\frac{dy}{dx} = xy^2 ; y(1) = 1$

Soln:  $f(x,y) = xy^2$  and  $\frac{\partial f}{\partial y} = 2xy$  are continuous for  $y > 0$  (continuous on the rectangle  $(-\infty, \infty) \times (0, \infty)$ ). Since the initial condition at the point  $(1, 1)$  is contained in the rectangle, the initial value problem has a unique solution.

14)  $\frac{dy}{dx} = 3\sqrt{y} ; y(0) = 0$

Soln:  $\sqrt{y}$  is continuous in the neighborhood of  $(0, 0)$ , but  $\frac{df}{dx}/\frac{dy}{dx} = \frac{1}{3}y^{-2/3}$  is not continuous here, so from the theorem, we have existence but not uniqueness in some neighborhood of  $x = 0$ .

15)  $\frac{dy}{dx} = \sqrt{x-y} ; y(2) = 2$       since:  $x = y$

Soln: Since  $f(x,y)$  is continuous for  $x \geq y$  and  $\frac{\partial^2 f}{\partial x \partial y} = \frac{1}{2\sqrt{x-y}}$  is continuous only for  $x > y$ , we are guaranteed neither existence nor uniqueness of a solution to this I.V.P.

16)  $\frac{dy}{dx} = \sqrt{x-y} ; y(2) = 1$

Soln: Now, since our initial condition has  $x > y$ , we are guaranteed a solution to the I.V.P.

17)  $y \frac{dy}{dx} = x-1 ; y(0) = 1$

Soln:  $f(x,y) = \frac{x-1}{y}$  and  $\frac{\partial f}{\partial y} = -\frac{(x-1)}{y^2}$  are both continuous near  $(0, 1)$ , so the theorem guarantees both existence and uniqueness of a solution in some neighborhood of  $x = 0$ .

18)  $y \frac{dy}{dx} = x-1 ; y(1) = 0$

Soln: Neither  $f(x,y)$  nor  $\frac{\partial f}{\partial y}$  are continuous near  $(1, 0)$ , so we are guaranteed neither existence nor uniqueness.

20)  $\frac{dy}{dx} = x^2 - y^2 ; y(0) = 1$

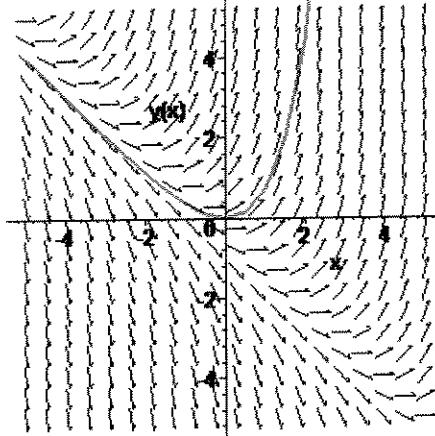
Soln: Both  $f(x,y)$  and  $\frac{\partial f}{\partial y} = -2y$  are continuous everywhere in the  $xy$ -plane, so the IVP has a unique solution.

In Problems 21 and 22, first use the method of Example 2 to construct a slope field for the given differential equation. Then sketch the solution curve corresponding to the given initial condition. Finally, use this solution curve to estimate the desired value of the solution  $y(x)$ .

21.  $y' = x + y$ ,  $y(0) = 0$ ;  $y(-4) = ?$

sln:

```
> with(DEtools):
> DEplot( diff(y(x),x) = x + y(x),y(x), x=-5..5, y=-5..5,
{[0,0]}, color = black, linecolor = blue, thickness = 2 );
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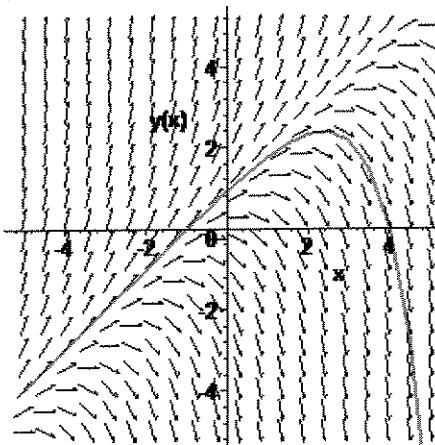


Here, we can see that  $y(-4) \approx 3$ .

22.  $y' = y - x$ ,  $y(4) = 0$ ;  $y(-4) = ?$

sln:

```
> DEplot( diff(y(x),x) = y(x)-x,y(x), x=-5..5, y=-5..5,
{[4,0]}, color = black, linecolor = blue, thickness = 2 );
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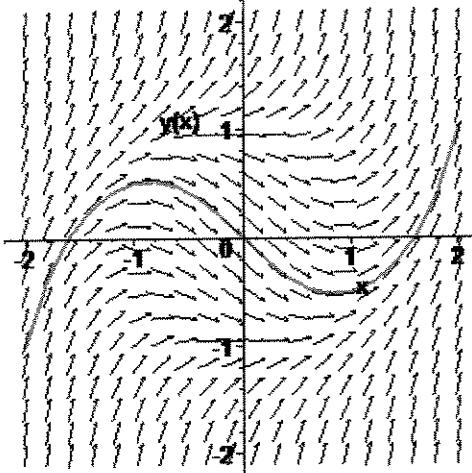
Here, we can see that  $y(-4) \approx -3$ .

Plot and print out a slope field for the given differential equation.

23.  $y' = x^2 + y^2 - 1$ ,  $y(0)=0$ ;  $y(2)=?$

soln:

```
>DEplot( diff(y(x),x) = x^2+y(x)^2-1,y(x), x=-2..2, y=-2..2,
  {[0,0]}, color = black, linecolor = blue, thickness = 2 );
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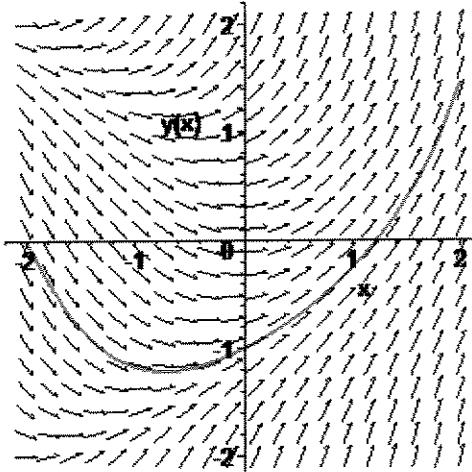


Tracing the curve in the figure,  $y(2) \approx 1$ .

24.  $y' = x + .5y^2 - 1$ ,  $y(-2)=0$ ;  $y(2)=?$

soln:

```
>DEplot( diff(y(x),x) = x+.5*y(x)^2-1,y(x), x=-2..2, y=-2..2,
  {[{-2,0}]}, color = black, linecolor = blue, thickness = 2 );
```



Tracing the curve in the figure,  $y(2) \approx 1.5$ .

## Section 1.4

## Homework Solutions

Problems 1-18, find general solutions of the differential equations.

2)  $\frac{dy}{dx} + 2xy^2 = 0$

$$\text{Solv: } \int \frac{dy}{y^2} = -\int 2x dx \Rightarrow \frac{-1}{y} = -x^2 - C \Rightarrow y(x) = \frac{1}{x^2 + C}$$

4)  $(1+x) \frac{dy}{dx} = 4y$

$$\text{Solv: } \int \frac{dy}{y} = \int \frac{4 dx}{(1+x)} \Rightarrow \ln y = 4 \ln(1+x) + \ln C \Rightarrow y(x) = C(1+x)^4$$

6)  $\frac{dy}{dx} = 3\sqrt{xy}$

$$\text{Solv: } \int \frac{dy}{\sqrt{y}} = \int 3\sqrt{x} dx \Rightarrow \frac{2\sqrt{y}}{2} = \frac{2x^{3/2}}{2} + \frac{2C}{2} \\ \sqrt{y} = x^{3/2} + C \Rightarrow y(x) = (x^{3/2} + C)^2$$

8)  $\frac{dy}{dx} = 2x \sec y$

$$\text{Solv: } \int \frac{1}{\sec y} dy = \int \cos(y) dy = \int 2x dx \\ \Rightarrow \sin(y) = x^2 + C \Rightarrow y(x) = \sin^{-1}(x^2 + C)$$

10)  $(1+x)^2 \frac{dy}{dx} = (1+y)^2$

$$\text{Solv: } \int \frac{dy}{(1+y)^2} = \int \frac{dx}{(1+x)^2} \Rightarrow -\frac{1}{1+y} = -\frac{1}{1+x} - C \\ \Rightarrow 1+y = \frac{1+x}{1+C(1+x)} \Rightarrow y(x) = \frac{x-C(1+x)}{1+C(1+x)}$$

12)  $yy' = x(y^2 + 1)$

$$\text{Solv: } \int \frac{y}{(y^2 + 1)} dy = \int x dx = \frac{1}{2} \ln(y^2 + 1) = \frac{1}{2} x^2 + \frac{1}{2} K.C$$

$$\Rightarrow \ln(y^2 + 1) = x^2 + K.C \Rightarrow y^2 + 1 = e^{(x^2 + K.C)}$$

$$\Rightarrow y(x) = \sqrt{e^{x^2} - 1}$$

problems 19-28, find explicit particular solutions of the initial value problems.

20)  $\frac{dy}{dx} = 3x^2(y^2 + 1)$ ;  $y(0) = 1$

solt:  $\int \frac{dy}{y^2+1} = \int 3x^2 dx$

$$\Rightarrow \tan^{-1} y = x^3 + C \quad y(x) = \tan(x^3 + C)$$

$$y(0) = 1 \Rightarrow 1 = \tan(C) \Rightarrow \tan^{-1}(1) = C \Rightarrow C = \pi/4$$

$$\therefore y(x) = \tan(x^3 + \pi/4)$$

24)  $(\tan x) \frac{dy}{dx} = y$ ;  $y(\pi/2) = \pi/2$

solt:  $\int \frac{dy}{y} = \int \frac{\sec x dx}{\tan x}$

$$\Rightarrow \ln y = \ln(\sin x) + \ln C, \quad y(x) = C \sin x$$

$$y(\pi/2) = C \sin(\pi/2) = \pi/2 \Rightarrow C = \pi/2,$$

$$\therefore y(x) = \frac{\pi}{2} \sin x$$

28)  $2\sqrt{x} \frac{dy}{dx} = \sec^2 y$ ;  $y(4) = \pi/4$

solt:  $\int \sec^2 y dy = \int \frac{dx}{2\sqrt{x}}$ ;  $\tan y = \sqrt{x} + C$   
 $\Rightarrow \sec^2 y = -2x^{-1/2} + C$ .

$$y(4) = \pi/4 \Rightarrow C = -1, \quad \text{so } y(x) = \tan^{-1}(\sqrt{x} - 1)$$

30) Solve the D.E.  $(\frac{dy}{dx})^2 = 4y$  to verify, see exercise Fig 1.4.5  
 Then determine the points  $(a, b)$  in the plane for which the I.V.P. has  
 a) no solution; b) exactly one; c) two if  $x > 0$ , finitely many otherwise.

solt:  $\frac{dy}{dx} = \pm 2\sqrt{y} \Rightarrow \int \frac{dy}{2\sqrt{y}} = \int dx, \quad (\because x > 0, \quad y = (x-a)^2)$   
 (this general soln gives the parabolas in Fig 1.4.5)  
 since  $y = (x-a)^2 \geq 0$ ,  $y$  is always non-negative.  
 Also, "note that"  $y(x) \equiv 0$  is a solution.

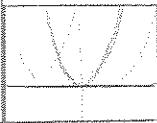


Fig 1.4.5

a) If  $b < 0$ , the I.V.P. has no solution.

b) If  $b \geq 0$ , there are many solution curves.

c) If  $a > 0$ , there are 2 solution curves through  $(a, b)$  intersecting to 2 points, i.e. 2 curves, one ascending and one descending.

- 31) Discuss the difference between the differential equations  $(\frac{dy}{dx})^2 = 4y$  and  $\frac{dy}{dx} = 2\sqrt{y}$ . Do they have the same solution curves? Why or why not? Determine the points  $(x_0, 0)$  in the plane for which the IVP  $y' = 2\sqrt{y}$ ,  $y(x_0) = b$  has a) no solution b) a unique solution c) infinitely many solutions.

$\Rightarrow$  The equation of motion of variable forces is the same (see prob 30)

In  $y' = \sqrt{y}$ ,  $y'$  can be either positive or negative, but in  $y' = 2\sqrt{y}$ ,  $y'$  must be nonnegative.

This implies that only the right half of each parabola  $y = (x-c)^2$  qualifies as a solution curve.

- 31
- a) If  $b < 0$ , no solution curve
  - b) If  $b > 0$ , a unique solution curve
  - c) If  $b = 0$ , infinitely many solution curves, because we can pick any  $c > a$  and define the solution  $y(x) = 0$  if  $x \leq c$ ,  $y(x) = (x-c)^2$  if  $x \geq c$ .

- 34) In a certain culture of bacteria, the number of bacteria increased sixfold in 10 h. How long did it take for the population to double?

Soln. The population growth rate ( $K$ ) is  $K = \ln(6)/10$ , so  $K = 0.17918$ .

After  $t$  hours, the population is given by

$$P(t) = P_0 e^{0.17918 t}$$

In order to solve for how long it takes for the population to double

$$2P = P e^{0.17918 t}$$

$$\ln(2) = \ln(e^{0.17918 t})$$

$$\ln(2) = 0.17918 t$$

$$\frac{\ln(2)}{0.17918} = t$$

$$\therefore t = 3.87 \text{ hours}$$

Problems 1-25, find general solutions of the differential equations if an I.C. is given, find the corresponding particular solution.

$$2) \quad y' - 2y = 3e^{2x} \quad y(0) = 0$$

$$\text{Soln: } 1 - P(x) = -2 \quad Q(x) = 3e^{2x}$$

$$2 - e^{\int P(x)dx} = e^{\int -2dx} = e^{-2x}$$

$$3 - \int Q(x)e^{\int P(x)dx} dx = \int 3e^{2x}e^{-2x} dx = 3x + C$$

$$\text{General Solution} \Rightarrow y(x) = e^{2x}(3x + C)$$

$$y(0) = e^0(3 \cdot 0 + C) = 0 \Rightarrow C = 0$$

$$\text{Particular solution} \Rightarrow y(x) = 3xe^{2x}$$

$$6) \quad xy' + 5y = 7x^2 \quad y(2) = 5$$

$$\text{Soln: } 1 - P(x) = 5/x, \quad Q(x) = 7x \quad (y' + \frac{5}{x}y = 7x)$$

$$2 - e^{\int P(x)dx} = e^{\int \frac{5}{x}dx} = e^{5\ln x} = x^5$$

$$3 - \int Q(x)e^{\int P(x)dx} dx = \int 7x \cdot x^5 dx = x^7 + C$$

$$\text{General Solution} \Rightarrow y(x) = x^{\frac{1}{2}}(x^7 + C) = x^2 + \frac{C}{x^5}$$

$$y(2) = 2^2(2^7 + C) = 5 \Rightarrow C = 32$$

$$\text{Particular solution} \Rightarrow y(x) = x^2 + \frac{32}{x^5}$$

$$18) \quad xy' = 2y + x^3 \cos x \Rightarrow y' - \frac{2}{x}y = x^2 \cos x$$

$$\text{Soln: } 1 - P(x) = -\frac{2}{x}, \quad Q(x) = x^2 \cos x$$

$$2 - e^{\int P(x)dx} = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = \frac{1}{x^2}$$

$$3 - \int Q(x)e^{\int P(x)dx} dx = \int x^2 \cos x \cdot \frac{1}{x^2} dx = \int \cos x dx$$

$$\text{General Solution} \Rightarrow y(x) = x^2(\sin x + C)$$

20)  $y' = 1 + x + y - xy$ ,  $y(0) = 0$

Soln. 1.  $P(x) = -1 - x$ ,  $Q(x) = 1 - x$

2.  $e^{\int P(x) dx} = e^{-\int (1+x) dx} = e^{-(x+\frac{x^2}{2})}$

3.  $\int Q(x)e^{\int P(x) dx} dx = \int (1-x)e^{-(x+\frac{x^2}{2})} dx = -e^{-(x+\frac{x^2}{2})} + C$

General Solution  $\Rightarrow y(x) = e^{-(x+\frac{x^2}{2})}(-e^{-(x+\frac{x^2}{2})} + C)$

$y(x) = -1 + Ce^{-(x+\frac{x^2}{2})}$

$y(0) = 0 \Rightarrow$  Particular solution  $\Rightarrow y(x) = -1 + e^{-x+\frac{x^2}{2}}$

22)  $y' = 2xy + 3x^2 \exp(x^2)$   $y(0) = 5$

Soln. 1.  $P(x) = -2x$ ,  $Q(x) = 3x^2 e^{x^2}$

2.  $e^{\int P(x) dx} = e^{-\int 2x dx} = e^{-x^2}$

3.  $\int Q(x)e^{\int P(x) dx} dx = \int 3x^2 e^{x^2} e^{-x^2} dx = x^3 + C$

General Solution  $\Rightarrow y(x) = e^{x^2}(x^3 + C)$

$y(0) = 1 \Leftrightarrow C = 5 \Rightarrow C = 5$

Particular solution  $\Rightarrow y(x) = e^{x^2}(x^3 + 5)$

24)  $(x^2 + 4)y' + 3xy = x$   $y(0) = 1$

Soln. 1.  $P(x) = -\frac{3x}{x^2+4}$ ,  $Q(x) = \frac{1}{x^2+4}$

2.  $e^{\int P(x) dx} = e^{\int -\frac{3x}{x^2+4} dx} = (x^2+4)^{3/2}$

3.  $\int \frac{x}{(x^2+4)^{3/2}} (x^2+4)^{3/2} dx = \int x(x^2+4)^{-1/2} dx = \frac{1}{3}(x^2+4)^{3/2} + C$

General Solution  $\Rightarrow y(x) = \frac{x^2+4)^{3/2}}{\frac{1}{3} + C(x^2+4)^{3/2}} = \frac{x^2+4)^{3/2}}{\frac{1}{3} + C(x^2+4)^{3/2}} + C$

$y(0) = 1 \Rightarrow \frac{1}{3} + C(0+4)^{3/2} = 1 \Rightarrow C = \frac{16}{3}$

Particular solution  $\Rightarrow y(x) = \frac{1}{3} + \frac{16}{3}(x^2+4)^{-3/2}$

- 30) Express the solution of the IVP  $2x \frac{dy}{dx} = y + 2x \cos x$ ,  $y(0) = 0$  as an integral as in Example 3 of this section.

Soln: 1.  $P(x) = -\frac{1}{2}x$ ,  $Q(x) = \cos(x)$

2.  $\int P(x) dx = e^{-\int P(x) dx/2} = |x|^{-1/2} = x^{-1/2}$   
(since the initial value  $x=1$  is positive)

3.  $\int Q(x) e^{\int P(x) dx} dx = \int \cos(x) x^{-1/2} dx$

General solution  $\rightarrow x^{1/2} \left( \int_1^x \cos(u) u^{-1/2} du + C \right)$

Particular solution  $\rightarrow 0 = y(1) = 1(0+C) \Rightarrow C = 0$ .

$$y(x) = x^{1/2} \int_1^x \cos(u) u^{-1/2} du$$

- 32) a) Find constants A and B such that  $y_p(x) = A \sin x + B \cos x$  is a solution of  $dy/dx + y = 2 \sin x$ .  
 b) Use the result of part a and the method of problem 31 to find the general solution of  $dy/dx + y = 2 \sin x$ .  
 c) Solve the IVP  $dy/dx + y = 2 \sin x$ ,  $y(0) = 1$ .

a)  $y = A \sin x + B \cos x$ ,  $y' = A \cos x - B \sin x$   
 $y' + y = (A + B) \cos x + (A - B) \sin x = 2 \sin x$

From (1)  $A = -1$  and  $B = 1$   
 Thus  $y_p(x) = -\sin x + \cos x$

b) General solution to  $y' + y = 0$  is  $y(x) = Ce^{-x}$   
 Adding this to part a) gives  $y(x) = Ce^{-x} + \sin x - \cos x$ .

c) The I.C. gives  $C = 2 \Rightarrow y(x) = 2e^{-x} + \sin x - \cos x$ .

- 31) a) Show that  $y_c(x) = Ce^{-\int P(x) dx}$  is a general solution of  $dy/dx + P(x)y = 0$ .  
 b) Show that  $y_p(x) = e^{-\int P(x) dx} \left[ \int Q(x) e^{\int P(x) dx} dx \right]$  is a particular solution of  $dy/dx + P(x)y = Q(x)$

Ans. a)  $y'_c = Ce^{-\int P(x) dx} (-P) = -Py_c$ , so  $y'_c + Py_c = 0$ .

b)  $y'_p = (-P)e^{-\int P(x) dx} \left[ \int Q(x) e^{\int P(x) dx} dx \right] + e^{-\int P(x) dx} Qe^{\int P(x) dx}$   
 $= -Py_p + Q$