

1. Consider a predator-prey population consisting of the foxes and rabbits living in a certain forest. Initially, there are F_0 foxes and R_0 rabbits; after k months, there are F_k foxes and R_k rabbits. Assume that the transition from each month to the next is described by the equations

$$\begin{aligned} F_{k+1} &= .4F_k + .3R_k \\ R_{k+1} &= -rF_k + 1.2R_k \end{aligned} \quad (\vec{x}_{k+1} = A\vec{x}_k \Rightarrow \vec{x}_k = A^k \vec{x}_0)$$

a) Describe what each of the terms on the RHS of these 2 eqns represents.

SOLN: (p. 384 of your text)

$r > 0$ is the capture rate of rabbits by foxes.

$.4F_k = 40\%$ of foxes would survive each month

$.3R_k =$ growth in fox population due to food supply (the rabbits)

$1.2R_k =$ in the absence of foxes, rabbits increase by 20% each month.

(Make sure that all the models that we have discussed make physical sense to you... this is not a stochastic matrix. Make sure you can deal with those as well.)

b) Find the characteristic equation of the transition matrix, solve evals.

SOLN: $100\lambda^2 - 160\lambda + (48 + 30r) = 0 \Rightarrow \lambda = \frac{1}{10}(8 \pm \sqrt{16 - 30r})$

c) if $r = .5$, what is the long-term behavior of the system?

SOLN: $\lambda_1 = .9, \lambda_2 = .7$.

when $\lambda_1 = .9$, then $\begin{bmatrix} -.5 & .3 \\ -.5 & .3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ from $(A - \lambda I)v = 0, v_1 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

for $\lambda_2 = .7$, then $\begin{bmatrix} -.3 & .3 \\ -.5 & .5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

IMPORTANT \Rightarrow then from $A = PDP^{-1}$, we have (since $P = \begin{bmatrix} 3 & 1 \\ 5 & 1 \end{bmatrix}, D = \begin{bmatrix} .9 & 0 \\ 0 & .7 \end{bmatrix}, P^{-1} = \begin{bmatrix} -1/2 & 1/2 \\ 5/2 & -3/2 \end{bmatrix}$)

KEY IDEA $\Rightarrow \underline{A^k} = \underline{P} \underline{D^k} \underline{P}^{-1} \dots \begin{bmatrix} 3 & 1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} (.9)^k & 0 \\ 0 & (.7)^k \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 5/2 & -3/2 \end{bmatrix} \approx \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ as $k \rightarrow \infty$.

so $\begin{bmatrix} F_k \\ R_k \end{bmatrix} = A^k \begin{bmatrix} F_0 \\ R_0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} F_0 \\ R_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (from $\vec{x}_k = A^k \vec{x}_0$)

Thus F_k and R_k both approach zero as $k \rightarrow \infty$. Both species die.

d) if $r = .4$, what is the long term behavior of the system?

SOLN: As in ex 3 of your book, p. 384,

$\lambda_1 = 1, v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \lambda_2 = .6, v_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, PDP^{-1} = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & .6 \end{bmatrix} \begin{bmatrix} -1/2 & 3/4 \\ 1/2 & -1/4 \end{bmatrix}$

$A^k = \begin{bmatrix} -1/2 & 3/4 \\ -1 & 3/2 \end{bmatrix}$, Hence, $\vec{x}_k = A^k \vec{x}_0 = \frac{1}{4} \begin{bmatrix} 3R_0 - 2F_0 \\ 6R_0 - 4F_0 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

If $3R_0 > 2F_0$, then as k increases, the fox and rabbit populations approach a stable situation with twice as many rabbits as foxes.

2. Apply the eigenvalue method to find a general solution of the following system:

$$\begin{aligned}x_1' &= 5x_1 - 9x_2 \\x_2' &= 2x_1 - x_2\end{aligned}$$

SOLN: $\begin{bmatrix} 5 & -9 \\ 2 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 5-\lambda & -9 \\ 2 & -1-\lambda \end{bmatrix} \Rightarrow \begin{aligned} &-(5-\lambda)(1+\lambda) + 18 = 0 \\ &-(5-\lambda^2-\lambda+5\lambda) + 18 = 0 \\ &\lambda^2 - 4\lambda + 13 = 0 \end{aligned}$

$$\lambda = \frac{4}{2} \pm \frac{1}{2}\sqrt{16 - 4(13)} \Rightarrow 2 \pm 3i$$

For $\lambda = 2 + 3i$

$$\begin{bmatrix} 5-2-3i & -9 \\ 2 & -1-2-3i \end{bmatrix} = \begin{bmatrix} 3-3i & -9 \\ 2 & -3-3i \end{bmatrix} \quad (3-3i)v_1 - 9v_2 = 0 \Rightarrow v = \begin{bmatrix} 3 \\ 1-i \end{bmatrix}$$

Therefore, $e^{\lambda t} \vec{v} = e^{(2+3i)t} \begin{bmatrix} 3 \\ 1-i \end{bmatrix} = e^{2t} (\cos(3t) + i\sin(3t)) \begin{bmatrix} 3 \\ 1-i \end{bmatrix}$

or, written out,
$$= \begin{bmatrix} 3e^{2t} \cos(3t) + 3ie^{2t} \sin(3t) \\ (1-i)e^{2t} \cos(3t) + i(1-i)e^{2t} \sin(3t) \end{bmatrix}$$

$$\rightarrow (e^{2t} \cos(3t) - ie^{2t} \cos(3t) + ie^{2t} \sin(3t) + e^{2t} \sin(3t))$$

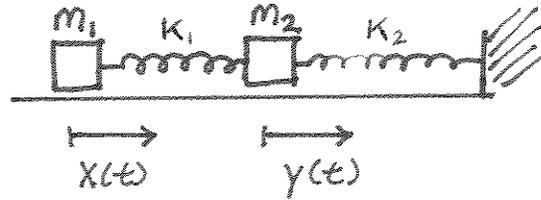
$$= \underbrace{\begin{bmatrix} 3e^{2t} \cos(3t) \\ e^{2t} \cos(3t) + e^{2t} \sin(3t) \end{bmatrix}}_{\text{Real part is a solution!}} + i \underbrace{\begin{bmatrix} 3e^{2t} \sin(3t) \\ -e^{2t} \cos(3t) + e^{2t} \sin(3t) \end{bmatrix}}_{\text{Imaginary part is another one!}}$$

Therefore,

$$\vec{x}_H(t) = e^{2t} \left[c_1 \begin{bmatrix} 3 \cos(3t) \\ \cos(3t) + \sin(3t) \end{bmatrix} + c_2 \begin{bmatrix} 3 \sin(3t) \\ -\cos(3t) + \sin(3t) \end{bmatrix} \right]$$

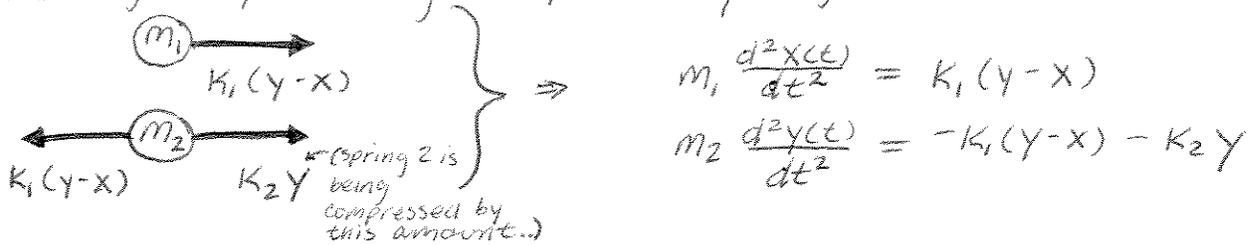
TIP: \uparrow this is an example that deals with finding the general solution of a system w/ imaginary eigenvalues, but you should be able to solve systems w/ real eigenvals as well (it's even easier!)

3. Consider the following configuration of springs, with positive displacements from equilibrium measured to THE RIGHT, as indicated.



a) Derive the system of second order differential equations which models this system.

SOLN: This configuration is reflected from the picture on page 431. We begin by drawing the free-body diagrams...



b) Assume that in appropriate units $m_1 = 2$, $m_2 = 2$, $K_1 = 4$, $K_2 = 6$. Show that in this case your system above reduces to:

$$\begin{aligned} x'' &= -2x + 2y \\ y'' &= 2x - 5y \end{aligned}$$

SOLN:
$$\begin{aligned} x'' &= \frac{K_1}{m_1} y - \frac{K_1}{m_1} x = \frac{4}{2} y - \frac{4}{2} x = -2x + 2y \\ y'' &= -\frac{K_1}{m_2} y + \frac{K_1}{m_2} x - \frac{K_2}{m_2} y = -2y + 2x - 3y = 2x - 5y \end{aligned}$$

c) Find the general solution to this unforced system.

SOLN: our A is $= \begin{bmatrix} -2 & 2 \\ 2 & -5 \end{bmatrix}$, we need its evals and evecs.

You should find $\lambda_1 = -1$, $v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\lambda_2 = -6$, $v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$
 Recall that the fundamental angular frequencies are given by $\lambda = -\omega^2$, so that our general solution is:

$$x(t) = (c_1 \cos(t) + c_2 \sin(t)) \begin{bmatrix} 2 \\ 1 \end{bmatrix} + (c_3 \cos(\sqrt{6}t) + c_4 \sin(\sqrt{6}t)) \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

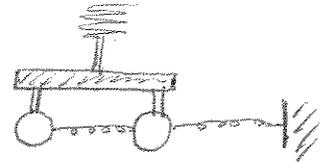
represents 1st natural mode of freq. represents 2nd natural mode of freq.

* Be prepared to explain how the masses interact in the different modes of frequency, based off of the eigenvector values.
 (Hint: see class Lecture notes or p. 432 of text)

3 d) Assuming that ω is not one of the natural frequencies for the problem in part c), find a particular solution to the FORCED system.
 (Draw what the forced component is doing to the system).

$$\begin{aligned}x'' &= -2x + 2y + \cos(\omega t) \\y'' &= 2x - 5y - \cos(\omega t)\end{aligned}$$

SOLN: $\vec{x}'' = A\vec{x} + \vec{f}$
 acting on both masses.



We try a particular solution of the form $x_p = \vec{c} \cos(\omega t)$
 $x_p' = -\vec{c}\omega \sin(\omega t)$
 $x_p'' = -\vec{c}\omega^2 \cos(\omega t)$ } let $\vec{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Plug back into the equation...

$$-\vec{c}\omega^2 \cos(\omega t) = A\vec{c} \cos(\omega t) + \vec{b} \cos(\omega t)$$

divide by the scalar function $\cos(\omega t)$ and reduce to

$$-\vec{b} = (A + \omega^2 I)\vec{c}$$

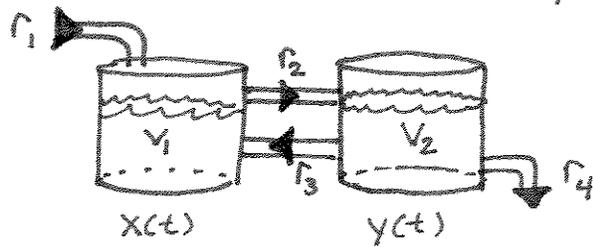
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$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 + \omega^2 & 2 \\ 2 & -5 + \omega^2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

You can now solve this for c_1 and c_2 either by using Cramer's rule or finding the inverse matrix,

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 - \omega^2 / (\omega^2 - 6)(\omega^2 - 1) \\ \omega^2 / (\omega^2 - 6)(\omega^2 - 1) \end{bmatrix}$$

4. Consider the following 2 tank configuration. In tank one there is a uniformly mixed volume of V_1 gallons, and pounds of solute $x(t)$. In tank two there is a mixed volume of V_2 gallons and pounds of solute $y(t)$. Water is pumped into tank one at a rate constant of r_1 gallons/minute from an outside source, and this water has a constant solute concentration of c_1 lbs./gal. Water is pumped from T_1 to T_2 at a constant rate of r_2 gal/min, from T_2 to T_1 at a constant rate r_3 gal/min, and out of the tank system at a constant rate r_4 gal/min.



a) What conditions on the rates r_1, r_2, r_3, r_4 are necessary to guarantee that the volumes V_1 and V_2 remain constant in time?

SOLN: For V_1 to remain constant, we need $r_1 + r_3 - r_2 = 0$
 For V_2 to remain constant, we need $r_2 - r_3 - r_4 = 0$

b) Write down, but do not solve, the system of 1st order differential equations which governs the process described above.

SOLN:

$$\frac{dx(t)}{dt} = r_1 c_1 - \frac{r_2 x(t)}{V_1} + \frac{r_3 y(t)}{V_2}$$

$$\frac{dy(t)}{dt} = \frac{r_2 x(t)}{V_1} - \frac{(r_3 + r_4) y(t)}{V_2}$$