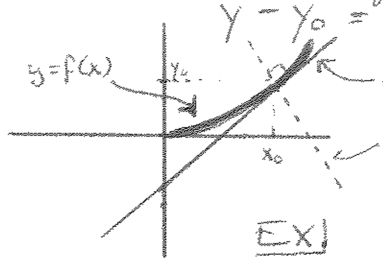
TUIZ #2 REVIEW

~ The slope & the tangent line ~

FORMULA: The equation of the tangent line

$$y - y_0 = m(x - x_0)$$



EX) $y = x^3 - 2$. Find the tangent line when $x = 2$.

SOLN: First, when $x = 2$, $y = 6$

At that point, $y'(x) = 3x^2$. $\Rightarrow$ $y'(2) =$ ____

The point-slope equation is found by 1) the slope and 2) the point slope equation: $y -$ ___ $=$ ___ $(x -$ ___ $)$.

SIMPLIFY... YOUR SOLN: ____________________.

These problems have applications for spacecraft controllers, baseball pitchers, monkeys, etc...

FORMULA: The equation for the secant line:

$$y - f(a) = \left[\frac{f(c) - f(a)}{c - a}\right](x - a)$$

Q: What happens as $c \to a$?

A: ____________________________________

~ The derivative of sin & cos ~

Q: What does the following differential equation describe?

$$y'' = -y$$

A: ____________________________________

ADDITION FORMULAS:

$$\sin(x \pm h) = \sin(x)\cos(h) \pm \cos(x)\sin(h)$$
$$\cos(x \pm h) = \cos(x)\cos(h) \mp \sin(x)\sin(h)$$

WORK THROUGH: using these, show $y' = \cos(x)$ for $y = \sin(x)$ by the standard method.

START: $$\frac{dy}{dx} = \lim \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sin(x + \Delta x) - \sin(x)}{\Delta x}$$

(HINT: $\frac{\cos(h) - 1}{h} = 0$ and $\frac{\sin(h)}{h} = 1$)

TUIZ #2 REVIEW

~ LIMITS ~

$\boxed{EX}$ Find $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$.

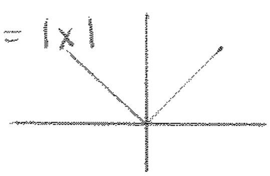
Soln: Substituting in $x = 3$ gives $\frac{0}{0}$. Meaningless!
Try $\frac{x^2 - 9}{x - 3} = \frac{(x-3)(x+3)}{(x-3)} = (x+3) \Rightarrow \lim_{x \to 3}$ gives $\underline{6}$.

~ CONTINUITY ~

- « A function is continuous if you can draw the graph without lifting your pen. »
- The Heaviside function  can be used to describe an electrical current that is switched on at $t = 0$. It has a jump discontinuity, but is still our friend.
- A function is "continuous" at a number "a" if:

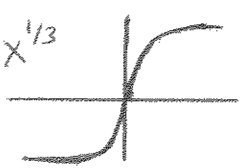
$$\lim_{x \to a} f(x) = f(a)$$

$\boxed{EX}$ $f(x) = |x|$



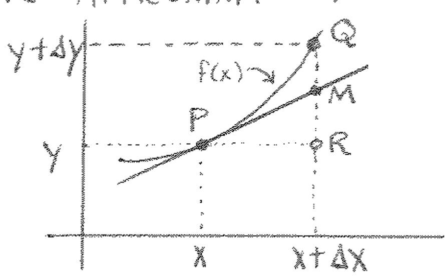
if f is differentiable at a, then it is continuous at a, but there are functions that are continuous, but not differentiable.

$\boxed{EX}$ $f(x) = x^{1/3}$



← Describe what is happening for this function at $x = 0$.
(um... maybe use your calculator to graph it for a better idea...)

~ LINEAR APPROXIMATION ~



Let $dx = \Delta x$
$dy = MR$
The slope of PM is $f'(x)$
(Recall: tan line is the instantaneous rate of change at x...)
$PM = dy/dx = f'(x)$
$\Rightarrow f'(x)dx = dy$

➡ For a function $y = f(x)$ whose derivative exists, the DIFFERENTIAL of y, dy, is $dy = f'(x)dx$.

As $dx \to 0$, $dy \to \Delta y$.
So for small nonzero values of dx,
$dy \approx \Delta y$, or $\Delta y = f'(x)dx$.

HEY YOU: IDENTIFY Δy and dy on the above graph...

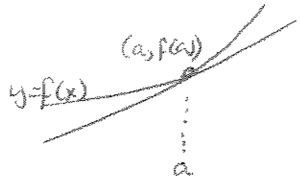
QUIZ # 2 REVIEW

LINEAR APPROXIMATION:

Let f be a function whose derivative exists. For small (nonzero) Δx, $dy \approx \Delta y$, and

$$f(x+\Delta x) \approx f(x) + dy = f(x) + f'(x)\,dx$$

KEY IDEA: Near the point of tangency, a curve lies very close to its tangent line.



An equation for the tangent line

$$y = f(a) + f'(a)(x-a)$$

So the linear approximation is from this idea

~ Max & Min Problems ~

KEY IDEA: Suppose the max or min occurs at a point x inside an interval where $f(x)$ and df/dx are defined. $\Rightarrow f'(x) = 0$.

Applications ... for a mass-spring system,

$$f(x) = \tfrac{1}{2}Kx^2 - mx$$

$$df/dx = Kx - m = 0. \Rightarrow Kx = m.$$

Physically, this is a balance of forces; the spring's force against the weight.

$\boxed{\text{EX}}$ (#23 in HW..) If h is fixed, show the maximum volume is $V = h(31 - \tfrac{1}{2}h)^2$. (Recall $l \le 62 - w - h$ ..)

$$V = l \cdot w \cdot h = (62 - w - h)wh$$

$$\frac{dV}{dh} = \underline{\qquad\qquad\qquad}$$

Solve for h .. $= \underline{\qquad\qquad\qquad}$

How do we show $V = h^3$ is the max volume?

$\boxed{\text{EX}}$ (# 46 in HW...) Find the right circular cylinder of largest volume that fits in a sphere of radius 1.

~ Second Derivatives ~

- A function with $f''(x) > 0$ is concave up.
  " " " $f''(x) < 0$ " concave down.

- An increasing population means $f'(x) > 0$.
  An increasing growth rate means $f''(x) > 0$.
  (What is the difference between these statements ?? . .)

- When $f'(x) = 0$ and $f''(x) > 0$, there is a local minimum at x.
  " " " " " $f''(x) < 0$, " " " " maximum " "

$\boxed{\text{EX}}$ Use these to find the max or min of

$$f(x) = x^3 - x^2$$

SOLN: $f'(x) =$ ________ and $f''(x) =$ ________

solve for $f'(x) = 0$. Then evaluate here . . .

~centered differences~

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} \quad \text{and} \quad \frac{f(x) - f(x-\Delta x)}{\Delta x}$$

What happens if we average these two averages?

$$\frac{1}{2}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x} + \frac{f(x) - f(x-\Delta x)}{\Delta x}\right] = \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x}$$

"CENTERED DIFFERENCE"

Q: Which is most accurate; centered, forward, or backward difference?

Q: What is the second difference scheme?

Hint: $\dfrac{\Delta f'(x)}{\Delta x}$ . . .

Q: What happens as $\Delta x \to 0$?

QUADRATIC APPROXIMATION:

(HINT: IMPORTANT!) $f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2$

(What happens if you add a third term? A fourth term? Infinite??)

[Ex] Write down the quadratic $f(0) + f'(0)x + \frac{1}{2}f''(0)(x)^2$ for $f(x) = \cos(x) + \sin(x)$.

SOLN:

[Ex] Guess the third order approximation
$$f(\Delta x) \approx f(0) + f'(0)\Delta x + \tfrac{1}{2} f''(0)\Delta x^2 + \_\_\_\_\_\_$$

[Ex] Find $f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2$ around $a=1$ for $f(x) = 1 + x + x^2$.

SOLN:

~ Iterations ~

- Iterating a function means applying a function repeatedly, using the output from one iteration as input to the next. Iterative methods are used to produce approximate numerical solutions to certain mathematical problems.

$\boxed{\text{Ex}}$ $x_{n+1} = F(x_n) = \frac{1}{2}x_n + 4$

START at $x_0 = 0$
⋮

START at $x_0 = 12$
⋮

When $x^* = F(x^*)$, x^* is a ____ ____.

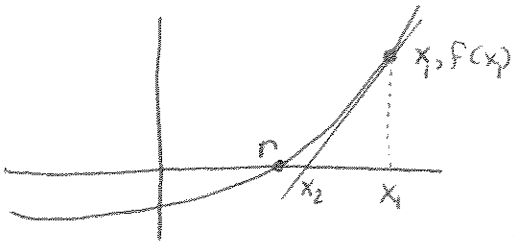
What makes this point attractive? Repulsive?

$$\frac{dF}{dx}(x^*) \ \boxed{?} \ 1.$$

Using the 1st order Taylor expansion, derive this result.

~ Newton's Method ~

* If f is a polynomial of degree 5 or higher, there is no formula to find the roots. Likewise, there is no such formula to find the exact roots of the transcendental equation $\cos(x) = x$.
(But we can find an approximate solution!)



- We seek r
- We start w/ an approximation, x_1 (By Guessing)
- Consider the tangent line to $y = f(x_1)$ and find the x-intercept, x_2...

How does our linear approximation tie in?

$$f(x) - f(x_1) = f'(x_1)(x - x_1)$$

[KNOW THE STEPS] $$\Rightarrow x - x_1 = \frac{-f(x_1)}{f'(x_1)}$$

⋮ iterate...

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$ ($\Rightarrow F(x_n) = x_{n+1}$. Looking for the FIXED POINT. F.P. of $F(x_n)$ is the ROOT of $f(x)$ as $\lim_{n\to\infty} x_n$...)

~ Mean Value Theorem ~

$\boxed{\text{EX}}$ Prove $f(x)$ is increasing when its slope is positive

(MATH SPEAK: $f'(c) > 0 \ \forall c \Rightarrow f(b) > f(a) \ \forall$ points $a, b \mid b > a$.)

Proof:

By the Mean Value theorem,

$$f(b) - f(a) = \underbrace{f'(c)}_{\text{Given } f'(c)>0\text{, the R.H.S. is positive.}}\underbrace{(b-a)}_{\text{Given } b>a\text{, this is positive}}$$

$$\Rightarrow f(b) - f(a) > 0.$$

$$f(b) > f(a)$$

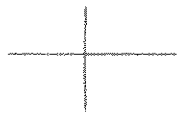
$\therefore$ A function with a positive slope is increasing.

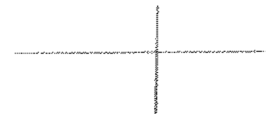
QUIZ # 2 REVIEW

~ The Chain Rule ~

Draw the graphs of:

$\cos(x^2)$ $\qquad$ $(\cos(x))^2$ $\qquad$ $\cos(\cos(x))$





These are not the same. WHY?

- Q: What is a composition of two functions?

CHAIN RULE: Suppose $g(x)$ has a derivative at x and $f(y)$ has a derivative at $y = g(x)$.
Then the derivative of $z = f(g(x))$ is

$$\frac{dz}{dx} = \frac{dz}{dy}\cdot\frac{dy}{dx} = f'(g(x))\cdot g'(x)$$

The slope of the composition $\uparrow$ is the product of the slopes of 2 functions... $\uparrow$

$\boxed{EX}$ $z = f(y) = y^n$. Find $\frac{d}{dx}[f(g(x))] = [g(x)]^n$

$\boxed{EX}$ $f = \sqrt{1+\sin t}$. Find df/dt.

$\boxed{EX}$ $f = \sin(\sqrt{1-x})$. Find df/dt. [Hint: triple chain rule!]

NOTE: It is important to know how to take the derivative. You will be accountable for:

- Addition rule
- product rule
- quotient rule
- power rule
- chain rule
- implicit differentiation

So review your favorite homework problems, and know where and when to use these rules!!

§4.2 Related Rates

- Suppose x and y are related to each other by $y = 2x$. If both variables are changing with respect to time, then their rates of change will also be related.

simply put: x & y are related $y = 2x$ ➡ The rates of change of x & y are related $\frac{dy}{dt} = 2\frac{dx}{dt}$

- You may be asked to create a mathematical model...

EX: A pebble is dropped into a calm pool of water, causing ripples in the form of concentric circles. The radius r of the outer ripple is increasing at a constant rate of 1 foot per second. When the radius is 4 feet, at what rate is the total area A of the disturbed water changing?

Hint: The variables r and A are related by $A = \pi r^2$.

Equation: $A = \pi r^2$

Given rate: $dr/dt = 1$ when $r = 4$

⇒ You: Find →: dA/dt when $r = 4$

Steps:
1) Write the original equation __________
2) Differentiate w.r.t. "t" __________
3) Apply chain rule __________
4) When $r = 4$ and $dr/dt = 1$, you have __________ $\frac{ft}{sec.}$

☆ Notice that the radius changes at a constant rate ($dr/dt = 1$) but the area changes at a nonconstant rate.

When $r = 1$ ft, $dA/dt =$ ______

when $r = 2$ ft, $dA/dt =$ ______

when $r = 3$ ft, $dA/dt =$ ______

Guidelines for related rates:
1- Identify given quantities, and quantities to be determined.
2- Write down an equation that relates the variables.
3- Use the chain rule to differentiate both sides of the equation w.r.t. time.
4- Substitute all known variables and rates of change. Then solve for the required rate of change.

Mathematical models for some rates of change...

VERBAL STATEMENT	MATHEMATICAL MODEL
• The velocity of a car after traveling 1 hour is 50 miles per hour.	x = distance traveled $dx/dt = 50$ when $t = 1$
• Water is being pumped into a swimming pool at a rate of 10 cubic feet per minute.	V = volume of pool water $\frac{dV}{dt} = 10\ \frac{ft^3}{min}$

Now you try...

• A population of bacteria is increasing at the rate of 2000 per hour.	____ = ________ $\frac{d}{dt}$ = ________
• Revenue is increasing at the rate of \$4000 per month	____ = ________ $\frac{d}{dt}$ = ________

FORMULAS

Sphere	Cylinder	Cone
$V = \frac{4}{3}\pi r^3$	$V = \pi r^2 h$	$V = \frac{1}{3}\pi r^2 h$
$A = 4\pi r^2$		

QUADRATIC FORMULA: $ax^2 + bx + c = 0$ $\Rightarrow$ $x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$