§5.1 (1) ~ The idea of the integral ~ (Handout)

KEY IDEA: Integration is about adding up infinitely many things.

- The goal of the next few sections is to relate the integral to the area under a curve, understand the Fundamental Theorem of Calculus, and to define "antiderivative", "indefinite", and "definite integrals."
- We begin by taking an algebraic approach (see §1.2 - Calc. without limits...)

$\boxed{\text{Ex}}$ Let $f_0, f_1, f_2, f_3, f_4 = 0, 2, 6, 12, 20$

Recall that in §1.2, we found $v_j = f_j - f_{j-1}$.

This means that

$$f_1 - f_0 = v_1 \Rightarrow 2 - 0 = 2$$
$$f_2 - f_1 = v_2 \Rightarrow 6 - 2 = 4$$
$$f_3 - f_2 = v_3 \Rightarrow 12 - 6 = 6$$
$$f_4 - f_3 = v_4 \Rightarrow 20 - 12 = 8$$

For simplicity, we can write this as:

f:	f_0		f_1		f_2		f_3		f_4
v:		v_1		v_2		v_3		v_4	

KEY → v's are the <u>differences</u> of f's...

(Note that the index of f starts at 0, whereas the index of v starts at 1.)

$\therefore v_1, v_2, v_3, v_4 = 2, 4, 6, 8$

- Now what if we want to go in the OTHER DIRECTION?

(That is to say, find the f's from the v's...)

$$v_1 = 2 \quad \leftarrow \text{That is } f_1 - f_0$$
$$v_2 + v_1 = 4 + 2 = 6 \quad \leftarrow \text{That is } f_2 - f_0$$
$$v_3 + v_2 + v_1 = 6 + 4 + 2 = 12 \quad \leftarrow \text{That is } f_3 - f_0$$
$$v_4 + v_3 + v_2 + v_1 = 8 + 6 + 4 + 2 = 20 \quad \leftarrow \text{That is } f_4 - f_0$$

↳ Let's write this last one out explicitly for clarity...

$$(f_4 - f_3) + (f_3 - f_2) + (f_2 - f_1) + (f_1 - f_0) = f_4 - f_0$$

cancel cancel cancel

KEY → So the <u>sum</u> of the v's gives us ... $\displaystyle\sum_{j=1}^{n} v_j = f_n - f_0$

(This will lead us to the Fundamental Theorem, $\int_a^b v(x)\,dx = f(b) - f(a)$ but for now, we are missing limits!)

§ 5.1 (2) (Handout)

$\boxed{EX}$ Now let's say $V_1 = V_2 = V_3 = V_4 = 3$ and $f_0 = 5$.

Lets find the f_j's. We do this by...

STARTING HERE ➡

$$\begin{array}{ccccccccc} f_0 & & f_1 & & f_2 & & f_3 & & f_4 \\ & V_1 = 3 & & V_2 = 3 & & V_3 = 3 & & V_4 = 3 & \end{array}$$

$f_1 - f_0 = V_1$

$f_1 - 5 = 3 \Rightarrow f_1 = 5 + 3 = 8$. Now we know f_1 and V_2...

$f_2 - f_1 = V_2$

$f_2 - 8 = 3 \Rightarrow f_2 = 11$

So $f_0 = 5$, $f_1 = 8$, $f_2 = 11$, $f_3 =$ ______, $f_4 =$ ______. HW 1

Now watch the pattern emerge...

$$\sum_{j=1}^{1} V_j = V_1 = f_1 - f_0 = 8 - 5 = 3$$

$$\sum_{j=1}^{2} V_j = V_1 + V_2 = f_2 - f_0 = 11 - 5 = 6$$

YOU DO IT → $\sum_{j=1}^{3} V_j = \qquad = f_3 - f_0 =$ ______ $=$ ______ HW 2

$\sum_{j=1}^{4} V_j = \qquad = f_4 - f_0 =$ ______ $=$ ______ .

* In the first part of the example, you work with differences.

In the second part, you are working with sums.

$\boxed{\# 5.1.1}$

With $v = 1, 2, 4, 8$, the formula for v is: ______ . HW 3

Hint: How do we find this? Notice $V_1 = 1 = 2^0$ ← index is 1, power is 1 - 1

$V_2 = 2 = 2^1$ ← index is 2, power is 2 - 1

$V_3 = 4 = 2^2$ ← index is 3, power is 3 - 1

$V_4 = 8 = 2^3$ ← index is 4, power is 4 - 1

Find f_1, f_2, f_3, f_4 starting from $f_0 = 0$. What is f_7? HW 4

§5.1 (3) (Handout)

$\boxed{\#5.1.2}$

The same $v = 1, 2, 4, 8$ are now the differences between $f = 1, 2, 4, 8, 16$. This means $f_0 = 1$ and $f_j = 2^j$.

a) Check that $2^5 - 2^4 = v_5$. HW5

b) What is $1 + 2 + 4 + 8 + 16$? HW6

Now graph v and f for HW4 and HW5.

What is the "step size" of all the previous problems? ________

CONCEPTS...

- In algebra, the difference $f_j - f_{j-1}$ is v_j.
  Conversely, the sum of the v's is $f_n - f_0$.
- In calculus, the derivative of $f(x)$ is $v(x)$. When we integrate the area under the curve, we get $f(x) - f(0)$.

→ Integration yields the area under the curve, $y = v(x)$.
It starts from rectangles with base Δx and heights $v(x)$,
The areas are (base)·(height) $= \Delta x \cdot v(x)$

KEY - As $\Delta x \to 0$, the area $v_1 \Delta x + v_2 \Delta x + \cdots v_n \Delta x$ becomes the integral of $v(x)$.

LOOKING FORWARD:

⇒ The sum $v_1 + \cdots v_n = f_n - f_0$ leads to the Fundamental Theorem: $\int_a^b v(x)\,dx = f(b) - f(a)$

DEFINITIONS:

⇒ The Indefinite integral is $f(x)$.

⇒ The definite integral is $f(b) - f(a)$.

RELATION TO §1.2:

⇒ Finding the area under the v-graph is the opposite of finding the slope of the f-graph.

§1.2 went this way..

$f_0 \quad f_1 \quad f_2 \quad f_3$

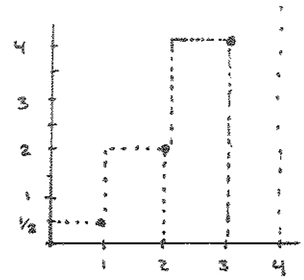
$v_1 \quad v_2 \quad v_3$

§5.1 went this way...

§ 5.1 (4) (WHERE WE ARE HEADED TOMORROW...) (Handout)

- Areas under $v(x) = y$ give $f(x)$...

Let's look at a function...



$\boxed{\text{Ex}}$ $y = v(x) = \frac{1}{2}x^2$

We can approximate by evaluating at the right-hand endpoint of rectangles ( ← Right-hand endpoint, base, $A = b \cdot h$)

"step size"

4 rectangles, $v(1) = \frac{1}{2}$, $v(2) = 2$, $v(3) = 9/2$, $v(4) = 8$, base $= 1$

★ The sum of the 4 areas is $1 \cdot \frac{1}{2} + 1 \cdot 2 + 1 \cdot \frac{9}{2} + 1 \cdot 8 = 15$

is 15 a good approximation to $f(4) - f(0)$?

Repeat the process for a stepsize of $\frac{1}{2}$ (implying 8 rectangles) for:

1) The height of the rectangle where the value of $v(x)$ is evaluated at the right-hand endpoint.

HW7

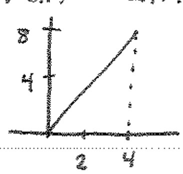
2) The height of the rectangle where the value of $v(x)$ is evaluated at the left-hand endpoint.

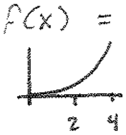
3) Looking forward: The actual area is found to be ______.

[Hint: $v(x) = \frac{1}{2}x^2 \Rightarrow \int_0^4 v(x)\,dx = f(4) - f(0)$]

$\boxed{\text{EX}}$ Relating areas under the curve... (easy)

The $v(x) = 2x$. The $f(x) = x^2$





HW8

hint: $\frac{1}{2} b \cdot h$

- The area under the curve from 0 to 2 is ______.

For 0 to 2, $f(2) - f(0)$ is ______.

- The area under the curve from 2 to 4 is ______.

For 2 to 4, $f(4) - f(2)$ is ______.