1 using the METHOD OF UNDETERMINED COEFFICIENTS, solve $y'' + 4y = x\sin 2x + 8$.

step 1 – $r^2 + 4 = 0 \Rightarrow r = \pm 2i$

$$y_c(x) = c_1 \cos 2x + c_2 \sin 2x$$

Now, guessing our particular,

$$x \sin 2x$$

$$\{x, 1\} \qquad \{\sin 2x, \cos 2x\}$$

$$= \{x\sin 2x,\ x\cos 2x,\ \sin 2x,\ \cos 2x\}$$

but these are already part of the complementary solution.

So multiply the entire family by x...

$$= \{x^2 \sin 2x,\ x^2\cos 2x,\ x\sin 2x,\ x\cos 2x\}$$

Also, the "family member" of 8 is $\{1\}$.

Our "guess" is therefore

$$y_p(x) = Ax^2\sin 2x + Bx^2\cos 2x + Cx\sin 2x + Dx\cos 2x + E$$

where A, B, C, D, and E are our coefficients, yet to be determined...

Substitute $y_p(x)$ back into the D.E.

$$[-8Bx + (2A - 4D)]\sin 2x + [8Ax + (2B + 4C)]\cos 2x + 4E$$

$$= x\sin 2x + 8$$

$$\begin{aligned} -8B &= 1 \\ -2A - 4D &= 0 \\ 8A &= 0 \\ 2B - 4C &= 0 \\ 4E &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} A &= 0 \\ B &= -1/8 \\ C &= 1/16 \\ D &= 0 \\ E &= 2 \end{aligned}$$

Thus, the general solution is

$$\underline{y(x) = c_1\cos 2x + c_2 \sin 2x - \tfrac{1}{8}x^2\cos 2x + \tfrac{1}{16}x\sin 2x + 2} \quad //$$

2. USING VARIATION OF PARAMETERS, solve the DE

$$y'' - 4y = \sinh(2x)$$

soln: The characteristic equation is

$$r^2 - 4 = 0 \Rightarrow r = \pm 2.$$

So the complementary solution is

$$y_c(x) = c_1 e^{2x} + c_2 e^{-2x}.$$

To find the particular solution, let $y_1(x) = e^{2x}$, $y_2(x) = e^{-2x}$.

(Know this $\Rightarrow$) $$y_p(x) = -y_1 \int \frac{y_2 f(x)}{W(x)}\,dx + y_2 \int \frac{y_1 f(x)}{W(x)}\,dx$$ [$\Leftarrow$ look back at your notes. Can you remember how we got this nice result?!]

The Wronskian is: $$W(x) = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = \underline{-4}$$

So the integrals are:

$$\int \frac{y_2 f(x)}{W(x)}\,dx = -\frac{1}{4}\int e^{-2x}\sinh(2x)\,dx$$

$$= -\frac{1}{4}\int e^{-2x}\left[\frac{e^{2x} - e^{-2x}}{2}\right]dx$$

$$= -\frac{1}{32} + \frac{x}{8} - \frac{e^{4x}}{32}$$

$$\int \frac{y_1 f(x)}{W(x)}\,dx = -\frac{1}{4}\int e^{2x}\sinh(2x)\,dx$$

$$= -\frac{x}{8} - \frac{e^{-4x}}{32}$$

Therefore, the general solution is:

$$y(x) = y_c(x) + y_p(x)$$

$$= c_1 e^{2x} + c_2 e^{-2x} + \frac{x}{8}\left(e^{2x} + e^{-2x}\right) - \frac{e^{2x}}{32} \quad //$$

3. Use Laplace transforms to solve the initial value problem

$$x'' + 4x = \sin(3t), \quad x(0) = x'(0) = 0$$

* Note: For the test, you will have Fig. 10.1.2 (p.572)

But → YOU NEED TO KNOW HOW TO SOLVE PARTIAL FRACTIONS!!!

soln: $\mathcal{L}\{x''\} + \mathcal{L}\{4x\} = \mathcal{L}\{\sin(3t)\}$

(Know where this step comes from!)

$$s^2 X(s) - s\,x(0) - x'(0) + 4X(s) = \frac{3}{s^2+9}$$

$$(s^2+4)X(s) = \frac{3}{s^2+9} + s\,x(0) + x'(0)$$

we know what these values are from our I.C.S...

$$X(s) = \frac{3}{(s^2+9)(s^2+4)}$$

$$\frac{3}{(s^2+9)(s^2+4)} = \frac{As+B}{(s^2+9)} + \frac{Cs+D}{(s^2+4)}$$

1st, clear the fractions to obtain

$$3 = (As+B)(s^2+4) + (Cs+D)(s^2+9)$$

The roots are $s = 3i$ and $s = 2i$.
Each complex root generates 2 equations, by equating the real & imaginary parts. Therefore, we will get 4 eqns in 4 unknowns.

$s = 3i$

$$3 = (3iA+B)(-9+4)$$

$$0i + 3 = -15iA - 5B \quad \Rightarrow \quad A = 0, \quad B = -3/5$$

$s = 2i$

$$3 = (2iC+D)(-4+9)$$

$$3 = 10iC + 5D \quad \Rightarrow \quad C = 0, \quad D = 3/5$$

$$X(s) = -\frac{3}{5}\frac{1}{(s^2+9)} + \frac{3}{5}\frac{1}{(s^2+4)}\left[\frac{2}{2}\right]$$

Here, I'm just multiplying by 1...

$$-\frac{1}{5}\left[\frac{3}{s^2+9}\right] + \frac{3}{10}\left[\frac{2}{s^2+4}\right]$$

to get it in this form.

so that taking the inverse Laplace Transform (using the table...)

$$x(t) = \underline{3/10 \sin(2t) - 1/5 \sin(3t)} \quad //$$

4. Find the transient motion and steady periodic oscillations of a damped mass and spring system with $m=1$, $c=2$, and $K=26$ under the influence of an external force $F(t) = 82\cos(4t)$, with $x(0)=6$ and $x'(0)=0$. (Ex. 6, p. 356)

Soln: We start with the IVP

$$x'' + 2x' + 26x = 82\cos 4t, \quad x(0)=6, \quad x'(0)=0.$$

The resulting motion of the mass will be

$$x(t) = x_{tr}(t) + x_{sp}(t)$$

We find the roots of the characteristic equation to be

$$r^2 + 2r + 26 \quad \Rightarrow \quad r = -1 \pm 5i$$

So that the complementary function is: $x_c(t) = e^{-t}(c_1\cos(5t) + c_2\sin(5t))$

We then make a guess for the particular solution, of the form $x(t) = A\cos(4t) + B\sin(4t)$

then $x'(t) = -4A\sin(4t) + 4B\cos(4t)$

and $x''(t) = -16A\cos(4t) - 16B\sin(4t)$

plugging these into the original D.E. and collecting $\sin(4t)$ and $\cos(4t)$ terms, we find $A=5$, $B=4$.

Our general solution is therefore:

$$x(t) = e^{-t}(c_1\cos(5t) + c_2\sin(5t)) + 5\cos 4t + 4\sin(4t)$$

Now use initial conditions to find the values of c_1 & c_2.

$c_1 = 1$, $c_2 = -3$

The transient behavior is $x_{tr}(t) = e^{-t}(\cos(5t) - 3\sin(5t))$

The steady periodic behavior is given by:

$$x_{sp}(t) = 5\cos(4t) + 4\sin(4t) = \sqrt{41}\cos(4t - \alpha).$$

(This is the behavior that is seen long term.)

5. Apply the convolution theorem to find the inverse Laplace transforms of the function $F(s) = \frac{1}{s(s^2+4)}$.

Soln: $\frac{1}{s(s^2+4)} = \frac{1}{2} \frac{1}{s} \cdot \frac{2}{s^2+4}$

From table,

$f(t)$	$F(s)$
1	$\frac{1}{s}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$
$\vdots$	

* see theorem 1 (p. 601).
Understand the convolution property.

$$f(t) = 1 * \frac{1}{2}\sin(2t)$$

our convolution is thus

$$= \int_0^t \frac{1}{2}\sin(2\tau)\,d\tau$$

$$= \frac{1}{4}(1-\cos 2t) \quad //$$

Note: we let $g(t) = \frac{1}{2}\sin(2t)$

$h(t) = 1$

so $g(t) * h(t) = \int_0^t g(\tau)\,h(t-\tau)\,d\tau$