Chapters 1 and 2 Review | Math 2250 | September 12, 2008

Name: *KEY*

1. Consider a differential equation which describes population growth,

$$\frac{dP}{dt} = aP^2 - BP \tag{1}$$

a) Find all equilibrium solutions to the system.

$$\frac{dP^*}{dt} = 0 = aP^*\left(P^* - \frac{B}{a}\right)$$

$$P^* = 0 \quad \text{and} \quad P^* = \frac{B}{a}$$

b) Suppose that the initial population is 1 thousand individuals and there are 2 thousand births per month and 1 thousand deaths per month. Write the corresponding differential equation for this scenario.

Birth term at time 0 : $aP(0)^2 = 2$

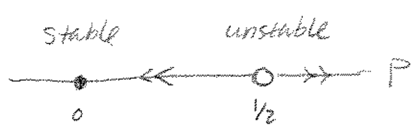
death term at time 0 : $BP(0) = 1$

Since $P(0) = 1$ (thousand), $a = 2$ and $B = 1$.

So $\frac{dP}{dt} = 2P^2 - P = 2P(P - \frac{1}{2})$

c) Sketch the phase line diagram for the equation you obtained in part (b). What are the equilibrium solutions? Which one is stable /unstable? Is there a threshold of population size to avoid extinction? Is there a limiting population size?

For part b, $P^* = 0$ and $P^* = 1/2$.



when $(P - 1/2) < 0$

$\Rightarrow P < 1/2, \Rightarrow \frac{dP}{dt} < 0.$

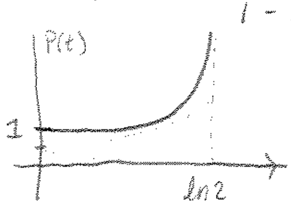
(arrows move to the left)

when $(P - 1/2) > 0$

$\Rightarrow P > 1/2, \quad \frac{dP}{dt} > 0$

(arrows move to the right)

The solution (check for yourself!!!) is $P(t) = \frac{1/2}{1 - Ae^t} \Rightarrow \frac{1}{2 - e^t}$.



As $t \to \ln(2)$, we get a population explosion. $P = 1/2$ is unstable. $P = 0$ (extinction) stable.

3. A brine tank holds 15000 gallons of continuously mixed liquid. Let $x(t)$ be the amount of salt (in pounds) in the tank at time t. Brine is flowing in and out at 150 gallons per hour and the concentration of the salt flowing in is 1 pound per 10 gallons of water. Find the differential equation for $x(t)$ and explain how you get this equation. Find the solution assuming that there is no salt in the water inside the tank initially. what is the limiting amount of salt as t approaches infinity?

Since the rate in = rate out, volume is constant.

$V = 15000$ gallon

$x(t)$ = amount of salt.

$c_i = \frac{1 \text{ lb}}{10 \text{ gals}} = .1$

$r_i = 150$ gal/hr

$$\frac{dx}{dt} = c_i r_i - \frac{x}{V} \cdot r_o$$

$$= 15 - \frac{150}{15{,}000} x$$

$$= 15 - .01x$$

solve by separation of variables.

$$\int \frac{dx}{15 - .01x} = \int dt + C$$

$$-\frac{1}{.01} \ln(15 - .01x) = t + C$$

$$\ln(15 - .01x) = -.01t + C$$

$$x(t) = 1500 + Ce^{-.01t}$$

$$\text{I.C.} \Rightarrow x(0) = 0 \Rightarrow C = -1500$$

$$x(t) = 1500 - 1500e^{-.01t}$$

as $t \to \infty$ the amount of salt approaches the limiting amount of 1500 lbs.

5. A driver involved in an accident claims he was going only 25 mph. When police tested his car, they found that when its brakes were applied at 25 mph, the car skidded only 45 feet before coming to a stop. But the driver's skid marks at the accident scene measured 210 feet. Assuming the same (constant) deceleration, determine the speed he was actually traveling just prior to the accident.

see HW soln. 1.2 # 44

6. a) Derive the solution of the logistic initial value problem $P' = kP(M - P), P(0) = P_0$. What happens as t approaches infinity? How does this depend on whether $0 < P_0 < M$ or $0 < M < P_0$?

See HW soln 2.1 # 32

b) Derive the solution of the extinction-explosion initial value problem $P' = kP(P - M), P(0) = P_0$. What happens as t approaches infinity? How does this depend on whether $0 < P_0 < M$ or $0 < M < P_0$?

see HW soln 2.1 # 33

7. The acceleration of a Maserati is proportional to the difference between 250 km/h and the velocity of this sports car. If this machine can accelerate from rest to 100 km/h in 10 s, how long will it take for the car to accelerate from rest to 200 kn/h?

see HW soln 2.3 # 1.