Section 3.3 Inverses

We use inverses all the time to solve problems in mathematics. An inverse takes you back to from whence you came.

Think of the equation $2x = 10$ how do we solve this equation?

In general, for real numbers we define the multiplicative inverse of $a \in R$ as $a^{-1} = \frac{1}{a}$ that it is a^{-1} is the number such that $a \cdot a^{-1} = 1$ where we call 1 the multiplicative identity since for any a, $a \cdot 1 = a$.

We have introduced the idea of the identity matrix for matrices the identity matrix is the multiplicative identity for the set of matrices $R^{n\times m}$.

$$AI = A$$
$$IA = A$$

So one question we might have is, is there an A^{-1} such that

$$A^{-1}A = I$$

And if so, when do we know it exists for a given matrix and how do we find it?

We want such a matrix because if we know A^{-1} we can solve $A\vec{x} = \vec{b}$ with simple matrix multiplication

$$\begin{aligned} A\vec{x} &= b \\ A^{-1}A\vec{x} &= A^{-1}b \\ I\vec{x} &= A^{-1}\vec{b} \\ \vec{x} &= A^{-1}\vec{b} \end{aligned}$$

Further, since a linear transformation on vectors is the same as matrix multiplication, this means we would have a way to find inverse linear transforms!

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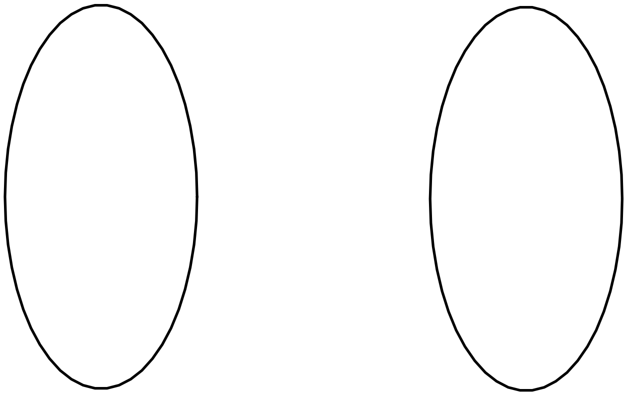
Just like with function inverses we will only have an inverse available for a linear transformation when it is 1-1 and onto.

A linear transformation $T: X \to Y$ will have an inverse transformation $T^{-1}: Y \to X$ if and only if it is 1-1 and onto. We difine T^{-1} to be the function such that if $T(x) = y$ then $T^{-1}(y) = x$



For a linear transformation on a space of vectors we usually transform $R^n \to R^m$. If we wish our linear transformation to have an inverse we will require that $n = m$.

Theorem: 3.21 A linear transformation T will have an inverse only when:

$T: R^n \to R^n, that\ is\ the\ dimension\ of\ the\ domain\ and\ codomain\ are\ the\ same.$

$Further\ if\ T\ has\ an\ inverse\ then\ T^{-1}\ is\ also\ a\ linear\ transformation.$

Proof:

Section 3.3 Inverses

If our linear transformation $T: R^n \to R^n$ then its matrix representation will be $n \times n$, it will be a square matrix.

We define the inverse matrix an $n \times n$ matrix A to be the matrix A^{-1} such that

$$A^{-1}A = I = AA^{-1}$$

Example: Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$ Show $B = A^{-1}$

Use the fact that $B = A^{-1}$ to solve the system: $x_1 + 3x_2 = 1$ and $2x_2 + 5x_2 = 2$

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One question we should immediately have is weather the inverse of a given matrix in unique (there is only one). We hope so!

Theorem 3.23: If A is an invertible matrix , A^{-1} is unique.

Proof:

Theorem 3.25: Let $A \ and \ B$ be invertable $n \times n$ matricies and C, D be $n \times m$ matricies.Then

$(a)\ A^{-1}\ is\ invertable, with\ \left(A^{-1}\right)^{-1} = A.$

$(b)\ \ AB\ is\ invertable, with\ (AB)^{-1} = B^{-1}A^{-1}$

$(c)\ \ IF\ AC = AD\ then\ C = D$

$(d)\ If\ \ AC = 0, then\ C = 0$

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Finding A^{-1}

In the last section we found that we could do row operations on matrices using elementary matrices. The elementary matrices were found by doing the desired row operation on the identity matrix.

Suppose we had k elementary matrices $E_1, E_2, E_3, \cdots, E_k$ which represented the k row operations that took a square matrix A to reduced row echelon form, further lets assume that this matrix had a pivot in every row and column in its rref form. This would mean that

$$E_k \cdots E_3 E_2 E_1 A = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} = I$$

Since matrix multiplication is associative we could instead multiply all of the $E's$ first $E_k \cdots E_3 E_2 E_1 = B$

This means $BA = I$ and since the inverse is unique it means $B = E_k \cdots E_3 E_2 E_1$ is the inverse.

Example: Let $A = \begin{bmatrix} 2 & 1/2 \\ 1 & 0 \end{bmatrix}$ find some elementary matrices which represent the required row operations to put A into rref form. Multiply the together and verify that the product is A^{-1}

Section 3.3 Inverses

All of those row operations are enough for a headache, then to have to multiply all of those matrices, especially for large one. There is a better way!

$$E_3\, E_2\, E_1 A = \; E_3\, E_2\, E_1 (I) A$$

We can sneak in an identity to the product of the elementary matrices and A if we like since $IA = A$. If we take and associative view of things.

$$(E_3\, (E_2\, (E_1 I)))A$$

*We first apply E_1 to I which performed the row operation $-2R_1 + R_2$ to I ,

*Then we multiply by E_2 which applies the operation $2R_1$ to the resulting matrix of $E_1 I$

*Then we multiply by E_3 which applies the operation $R_1 \leftrightarrow R_2$ to the result of $E_2 E_1 I$

This is exactly equivalent to just applying the row operations to the identity in the first place! So in order to cut the work in getting to A^{-1} all we have to do is apply the same row operations to the identity I as we do A to put A into reduced row echelon form. What ever we do to A we do to I at the same time, then when A hits rref, the identity I will have been transformed into A^{-1}!

> Given an invertible $n \times n$ matrix A you can find A^{-1} by augmenting A with the $n \times n$ identity matrix and perfeorming row operations until you reach reduced row echelon form for A. The identity matrix will have then been transformed into A^{-1}

$$\begin{tikzcd}
A & \left[\begin{array}{cccc|cccc} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & 0 & \cdots & 1 \end{array}\right] \arrow[l] \arrow[r] \arrow[d, "\text{RREF}"'] & I \\
I & \left[\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & 1 & \cdots & 0 & b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & b_{n1} & b_{n2} & \cdots & b_{nn} \end{array}\right] \arrow[l] \arrow[r] & A^{-1}
\end{tikzcd}$$

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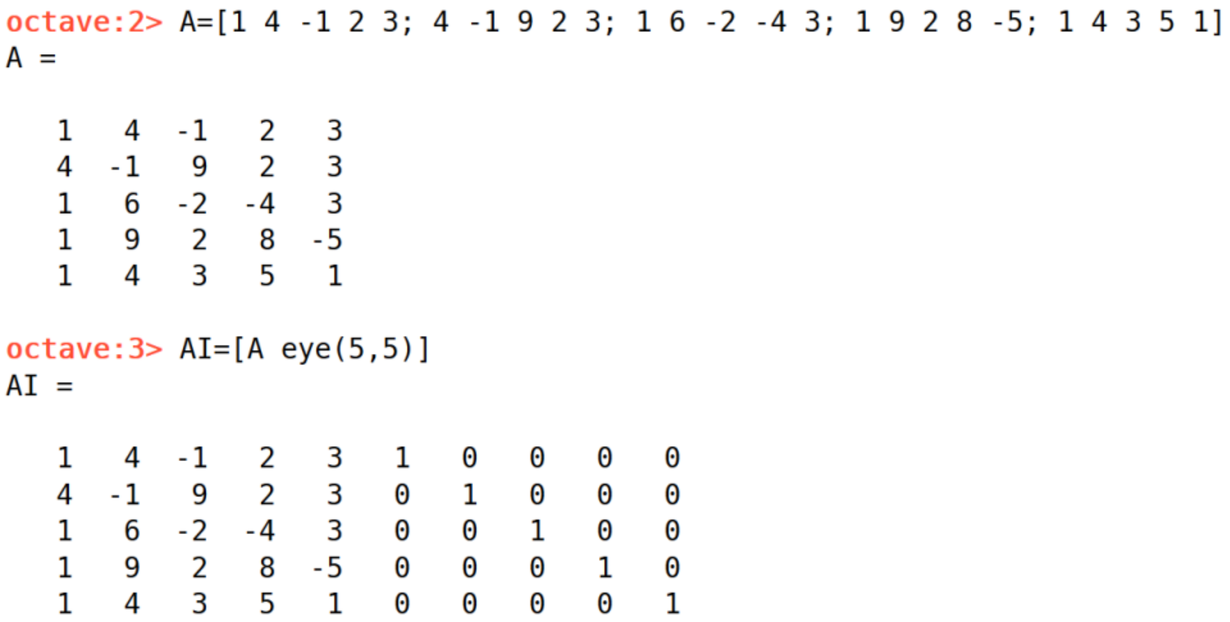
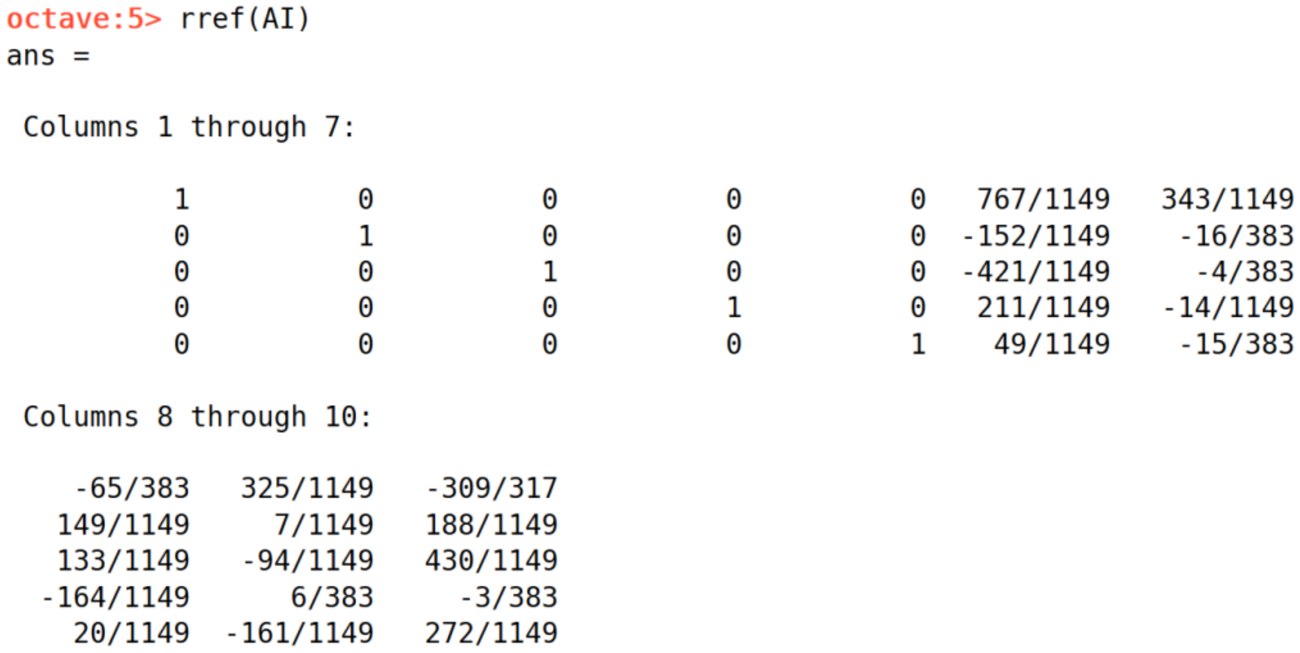
Example, Find A^{-1}, if it exists.

$$A = \begin{bmatrix} 1 & -2 & 1 \\ -3 & 7 & 6 \\ 2 & -3 & 0 \end{bmatrix}$$

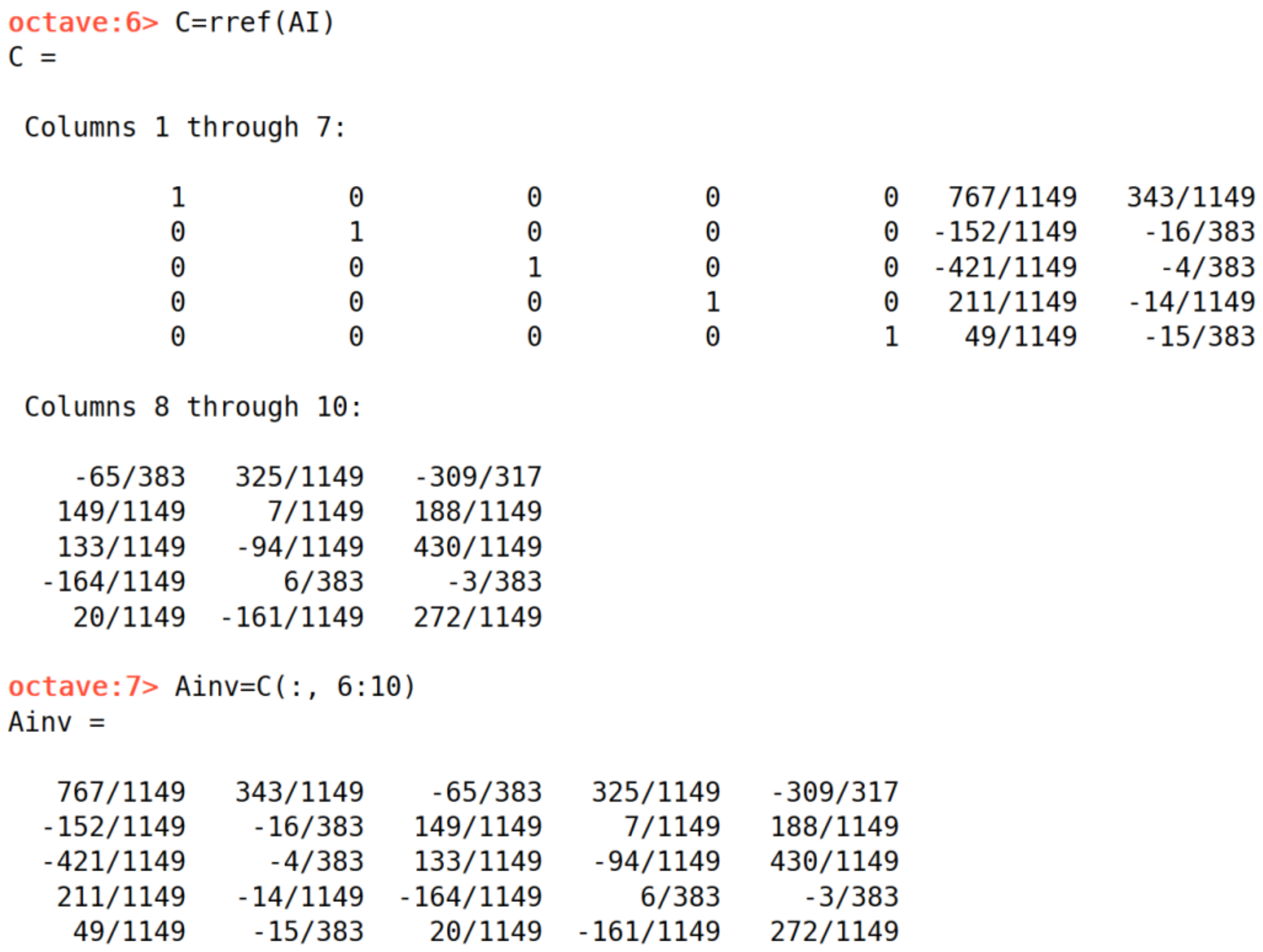
Section 3.3 Inverses



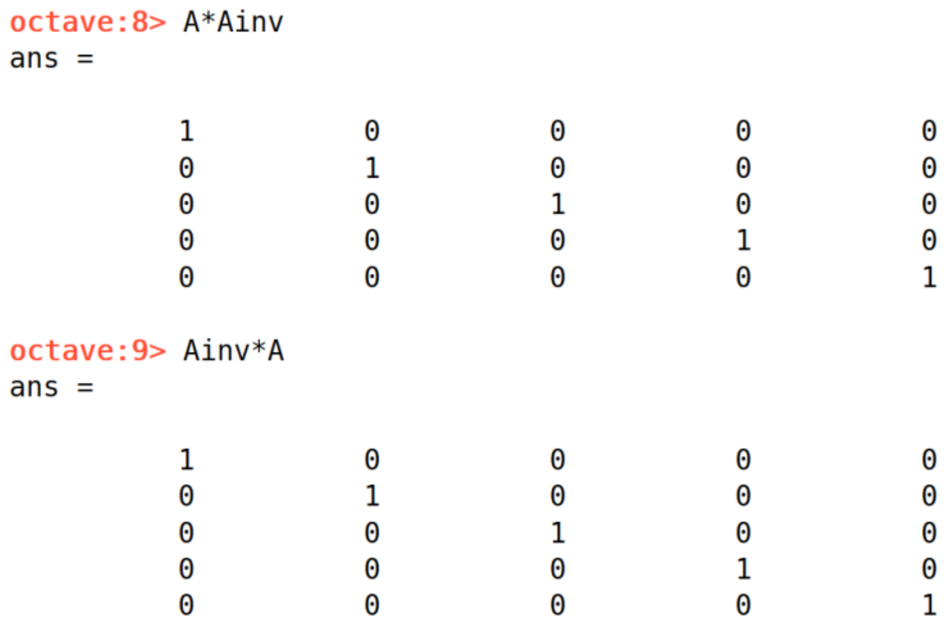
```
octave:2> A=[1 4 -1 2 3; 4 -1 9 2 3; 1 6 -2 -4 3; 1 9 2 8 -5; 1 4 3 5 1]
A =

   1   4  -1   2   3
   4  -1   9   2   3
   1   6  -2  -4   3
   1   9   2   8  -5
   1   4   3   5   1

octave:3> AI=[A eye(5,5)]
AI =

   1   4  -1   2   3   1   0   0   0   0
   4  -1   9   2   3   0   1   0   0   0
   1   6  -2  -4   3   0   0   1   0   0
   1   9   2   8  -5   0   0   0   1   0
   1   4   3   5   1   0   0   0   0   1
```



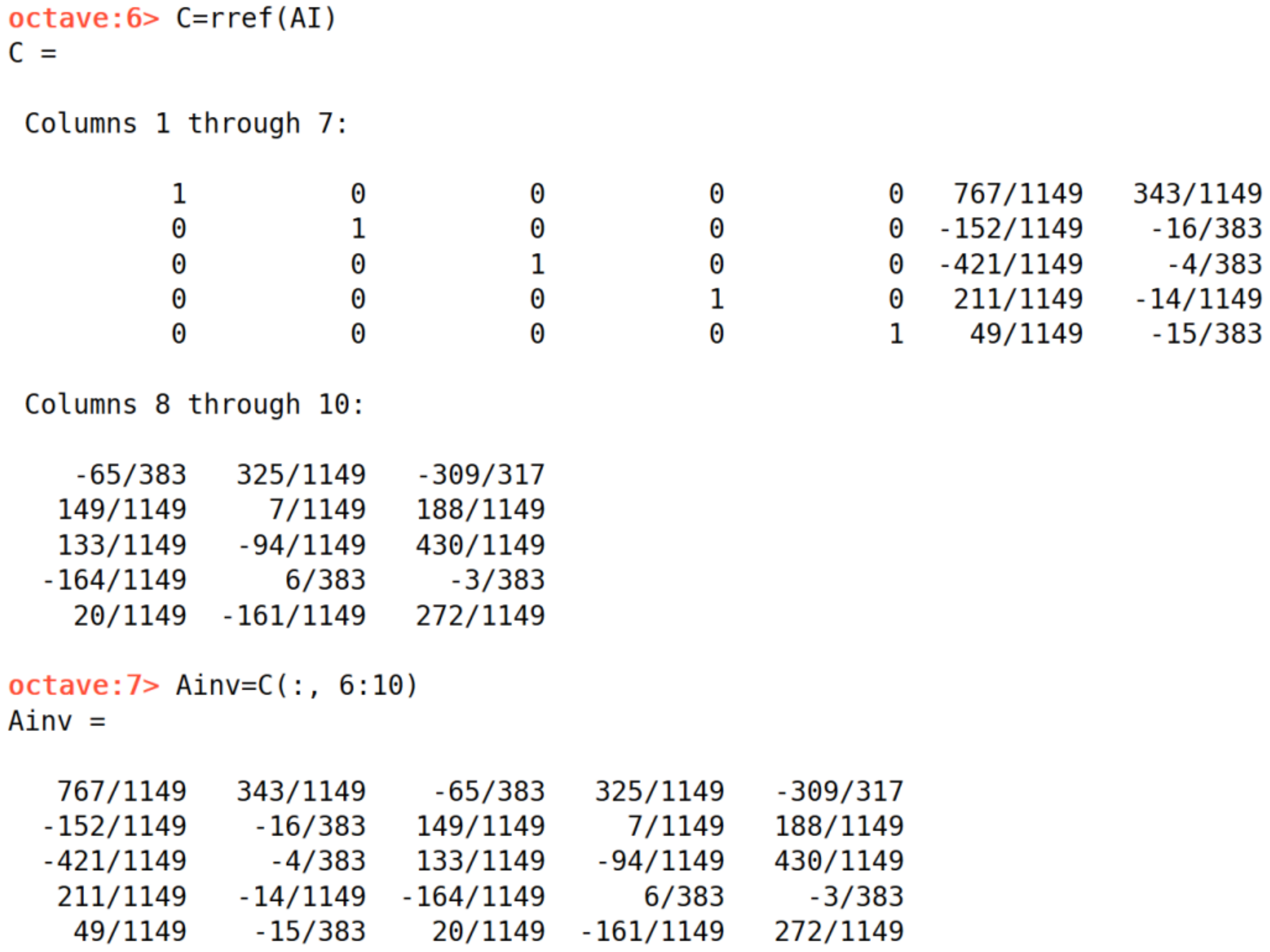
```
octave:5> rref(AI)
ans =

 Columns 1 through 7:

          1           0           0           0           0   767/1149   343/1149
          0           1           0           0           0  -152/1149    -16/383
          0           0           1           0           0  -421/1149     -4/383
          0           0           0           1           0   211/1149   -14/1149
          0           0           0           0           1    49/1149    -15/383

 Columns 8 through 10:

    -65/383   325/1149   -309/317
   149/1149     7/1149   188/1149
   133/1149   -94/1149   430/1149
  -164/1149      6/383     -3/383
    20/1149  -161/1149   272/1149
```



```
octave:6> C=rref(AI)
C =

 Columns 1 through 7:

          1           0           0           0           0    767/1149    343/1149
          0           1           0           0           0   -152/1149     -16/383
          0           0           1           0           0   -421/1149      -4/383
          0           0           0           1           0    211/1149    -14/1149
          0           0           0           0           1     49/1149     -15/383

 Columns 8 through 10:

     -65/383    325/1149    -309/317
    149/1149      7/1149    188/1149
    133/1149    -94/1149    430/1149
   -164/1149       6/383      -3/383
     20/1149   -161/1149    272/1149

octave:7> Ainv=C(:, 6:10)
Ainv =

    767/1149    343/1149     -65/383    325/1149    -309/317
   -152/1149     -16/383    149/1149      7/1149    188/1149
   -421/1149      -4/383    133/1149    -94/1149    430/1149
    211/1149    -14/1149   -164/1149       6/383      -3/383
     49/1149     -15/383     20/1149   -161/1149    272/1149
```

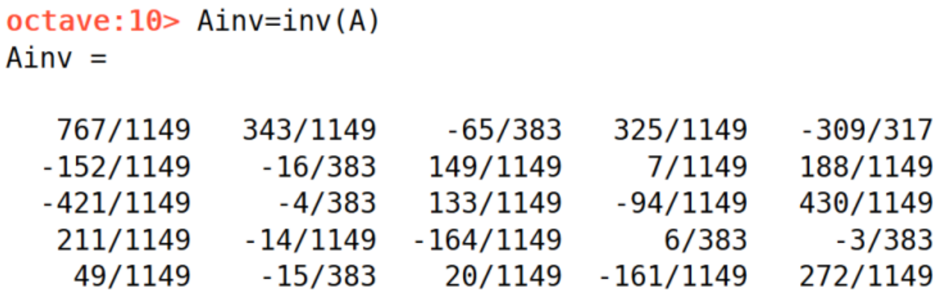


```
octave:8> A*Ainv
ans =

          1           0           0           0           0
          0           1           0           0           0
          0           0           1           0           0
          0           0           0           1           0
          0           0           0           0           1

octave:9> Ainv*A
ans =

          1           0           0           0           0
          0           1           0           0           0
          0           0           1           0           0
          0           0           0           1           0
          0           0           0           0           1
```



```
octave:6> C=rref(AI)
C =

 Columns 1 through 7:

           1           0           0           0           0    767/1149    343/1149
           0           1           0           0           0   -152/1149     -16/383
           0           0           1           0           0   -421/1149      -4/383
           0           0           0           1           0    211/1149    -14/1149
           0           0           0           0           1     49/1149     -15/383

 Columns 8 through 10:

     -65/383    325/1149   -309/317
    149/1149      7/1149   188/1149
    133/1149    -94/1149   430/1149
   -164/1149       6/383     -3/383
     20/1149   -161/1149   272/1149

octave:7> Ainv=C(:, 6:10)
Ainv =

    767/1149    343/1149     -65/383    325/1149    -309/317
   -152/1149     -16/383    149/1149      7/1149    188/1149
   -421/1149      -4/383    133/1149    -94/1149    430/1149
    211/1149    -14/1149   -164/1149       6/383      -3/383
     49/1149     -15/383     20/1149   -161/1149    272/1149
```

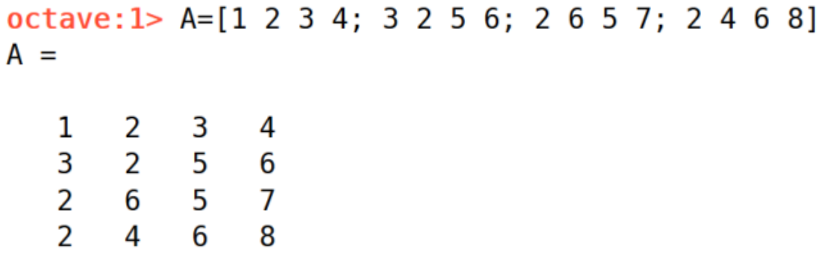
Or…..



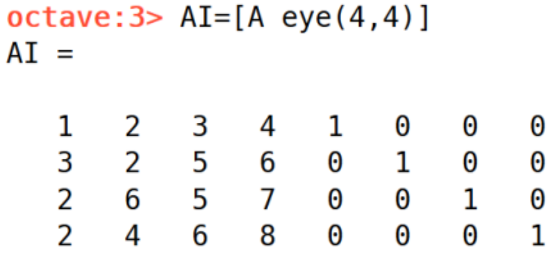
```
octave:10> Ainv=inv(A)
Ainv =

    767/1149    343/1149     -65/383    325/1149    -309/317
   -152/1149     -16/383    149/1149      7/1149    188/1149
   -421/1149      -4/383    133/1149    -94/1149    430/1149
    211/1149    -14/1149   -164/1149       6/383      -3/383
     49/1149     -15/383     20/1149   -161/1149    272/1149
```

Section 3.3 Inverses

When there is no inverse, you will get no solution



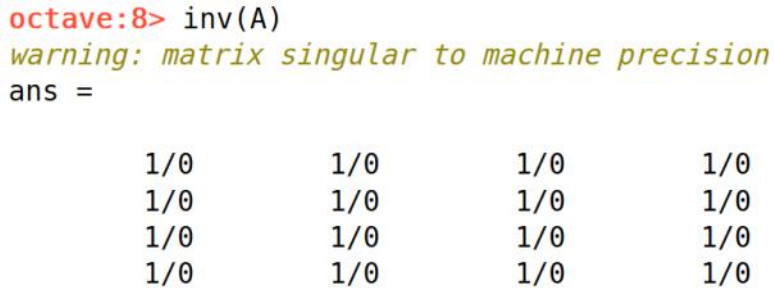
```
octave:1> A=[1 2 3 4; 3 2 5 6; 2 6 5 7; 2 4 6 8]
A =

   1   2   3   4
   3   2   5   6
   2   6   5   7
   2   4   6   8
```



```
octave:3> AI=[A eye(4,4)]
AI =

   1   2   3   4   1   0   0   0
   3   2   5   6   0   1   0   0
   2   6   5   7   0   0   1   0
   2   4   6   8   0   0   0   1
```

```
octave:4> rref(AI)
```

```
1          0          0       -1/3          0        2/3        1/3       -5/6
0          1          0        1/6          0      -1/12        1/3      -5/24
0          0          1        4/3          0       -1/6       -1/3       7/12
0          0          0          0          1          0          0       -1/2
```



```
octave:8> inv(A)
warning: matrix singular to machine precision
ans =

        1/0         1/0         1/0         1/0
        1/0         1/0         1/0         1/0
        1/0         1/0         1/0         1/0
        1/0         1/0         1/0         1/0
```

If a matrix does not have an inverse we call it **singular.**

Section 3.3 Inverses

Some nice formulas for nice matricies.

If a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in invertable then $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$A^{-1}A = \frac{1}{ad-ac}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-ac}\begin{bmatrix} ad-c & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

From this formula, we can tell A would not have an inverse if $ab - cd = 0$, we can't divide by zero!

If a diagonal matrix $A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$ in invertible then $A^{-1} = \begin{bmatrix} 1/a_{11} & 0 & 0 & \cdots & 0 \\ 0 & 1/a_{22} & 0 & \cdots & 0 \\ 0 & 0 & 1/a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1/a_{nn} \end{bmatrix}$

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} 1/a_{11} & 0 & 0 & \cdots & 0 \\ 0 & 1/a_{22} & 0 & \cdots & 0 \\ 0 & 0 & 1/a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1/a_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}/a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22}/a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33}/a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn}/a_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Here we could see that if any of the $a_{ii}'s$ are zero the diagonal matrix would not have an inverse.

Section 3.3 Inverses

Theorem 3.27 Let $S = \{\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n\}$ be a set of n vectors in R^n. Define the matrix $A = [\vec{a}_1 \ \vec{a}_2 \ \cdots \vec{a}_n]$, And let $T: R^n \to R^n$ be given hy $T(\vec{x}) = A\vec{x}$. Then the following are equivlent.

$(a)\ \ S\ spans\ R^n$
$(b)\ \ S\ \ is\ linearly\ independent$
$(c)\ Ax = b\ \ has\ a\ unique\ solution\ for\ all\ \vec{b} \in R^n$
$(d)\ T\ is\ onto$
$(e)\ T\ is\ one - to - one$
$(f)\ \boldsymbol{A\ is\ invertable}$