Section 2.3 Linear Independence

Linear independence is an important concept in linear algebra and sets of vectors which are linearly independent allow us to guarantee useful properties about those vectors. A set of vectors is linearly independent when no one vector in the set can be written as a linear combination of the others. In a sense it means it is not redundantly listed in the set. We can state this equivalently in the following way.

Let $\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \cdots \vec{u}_m\}$ be a set of vectors in R^n if the only solution to

$x_1\vec{u}_1 + x_2\vec{u}_2 + x_3\vec{u}_3 + \cdots + x_m\vec{u}_n = 0$ $\rightarrow$ $x_1\vec{v}_1 + x_2\vec{v} + \cdots + x_m\vec{v}_n = -x_3\vec{v}_3$

Is the trivial solution , $x_1 = x_2 = x_3, \cdots, x_m = 0$ then the set of vectors is **linearly independent,** if there are any non trivial (non-zero) solutions the set is called **linearly dependent**.

Example: Determine if the following set of vectors are linearly independent

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 5 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$v_1 \quad v_2 \quad v_3 \quad v_4$

Let's think about what this would mean in terms of a linear system.

$$x_1\begin{bmatrix} 1 \\ 2 \\ -3 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 4 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 1 \\ 3 \\ 5 \\ -1 \end{bmatrix} + x_4\begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} x_1 + 4x_2 + x_3 + 0x_4 &= 0 \\ 2x_1 - 2x_2 + 3x_3 + 2x_4 &= 0 \\ -3x_1 + x_2 + 5x_3 + x_4 &= 0 \\ x_1 + 0x_2 + x_3 + x_4 &= 0 \end{aligned} \Rightarrow x_1 = x_2 = x_3 = x_4 = 0$$

Section 2.3 Linear Independence

```
octave:32> u1=[1;2;-3;1]
u1 =

   1
   2
  -3
   1

octave:33> u2=[4;-2;1;0]
u2 =

   4
  -2
   1
   0

octave:34> u3=[1;3;5;-1]
u3 =

   1
   3
   5
  -1

octave:35> u4=[0;2;1;1]
u4 =

   0
   2
   1
   1
```

$$\left\{\begin{bmatrix}1\\2\\-3\\1\end{bmatrix},\begin{bmatrix}4\\-2\\1\\0\end{bmatrix},\begin{bmatrix}1\\3\\5\\-1\end{bmatrix},\begin{bmatrix}10\\-14\\-23\\6\end{bmatrix}\right\}$$

```
octave:36> A=[u1,u2,u3,u4,zeros(4,1)]
A =
```

$$\begin{array}{ccccc} u_1 & v_2 & v_3 & v_4 & \vec{0} \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 4 & 1 & 0 & 0 \\ 2 & -2 & 3 & 2 & 0 \\ -3 & 1 & 5 & 1 & 0 \\ 1 & 0 & -1 & 1 & 0 \end{array}\right]$$

```
octave:37> rref(A)
ans =
```

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$x_1 = 0$

$x_2 = 0$

$0x_1 + 0x_2 + x_3 + 0x_4 = 0$

$0x_1 + 0x_2 + 0x_3 + x_4 = 0$

Section 2.3 Linear Independence

Your turn! Grab your laptops and head over to octave-online.net (5 min in class excise)

Determine if the following set of vectors is linearly independent

$$\left\{\begin{bmatrix}1\\2\\-3\\1\end{bmatrix},\begin{bmatrix}4\\-2\\1\\0\end{bmatrix},\begin{bmatrix}1\\3\\5\\-1\end{bmatrix},\begin{bmatrix}10\\-14\\-23\\6\end{bmatrix}\right\}$$

```
octave:32> u1=[1;2;-3;1]
u1 =

   1
   2
  -3
   1

octave:33> u2=[4;-2;1;0]
u2 =

   4
  -2
   1
   0
```

```
octave:34> u3=[1;3;5;-1]
u3 =

   1
   3
   5
  -1

octave:38> u4=2*u1+3*u2-4*u3
u4 =

   10
  -14
  -23
    6
```

```
octave:39> A=[u1,u2,u3,u4,zeros(4,1)]
A =

    1    4    1   10    0
    2   -2    3  -14    0
   -3    1    5  -23    0
    1    0   -1    6    0

octave:40> rref(A)
ans =

    1    0    0    2   -0
    0    1    0    3    0
    0    0    1   -4    0
    0    0    0    0    0
```

$x_1 = +2t$

$x_1 \mp 2t = 0$

$x_2 \cancel{+} 3t = 0 \to x_2 = +3t$

$\to x_3 + 4t = 0 \to x = {}^{-}4t$

$\to x_4 = {}^{-}t$

Let's verify that $2t\vec{u}_1 + 3t\vec{u}_2 - 4t\vec{u}_3 = t\vec{u}_4$

$$\underline{2t}\begin{bmatrix}1\\2\\-3\\1\end{bmatrix} + \underline{3t}\begin{bmatrix}4\\-2\\1\\0\end{bmatrix} - \underline{4t}\begin{bmatrix}1\\3\\5\\-1\end{bmatrix} = \underline{t}\begin{bmatrix}10\\-14\\-23\\6\end{bmatrix}$$

d

$$\begin{bmatrix}2t\\4t\\-6t\\2t\end{bmatrix} + \begin{bmatrix}12t\\-6t\\3t\\0t\end{bmatrix} + \begin{bmatrix}-4t\\-12t\\-20t\\4t\end{bmatrix} = \begin{bmatrix}(2+12-4)t\\(4-6-12)t\\(-6+3-2)t\\(2+0+4)t\end{bmatrix} = \begin{bmatrix}10t\\-14t\\-23t\\6t\end{bmatrix}$$

Section 2.3 Linear Independence

Show that the vectors $\left\{ \begin{bmatrix}16\\2\\8\end{bmatrix}, \begin{bmatrix}22\\4\\4\end{bmatrix}, \begin{bmatrix}18\\0\\4\end{bmatrix}, \begin{bmatrix}18\\2\\6\end{bmatrix} \right\}$ are linearly dependent

```
octave:4> u1=[16;2;8]
u1 =

   16
    2
    8

octave:5> u2=[22;4;4]
u2 =

   22
    4
    4

octave:6> u3=[18;0;4]
u3 =

   18
    0
    4

octave:7> u4=[18;2;6]
u4 =

   18
    2
    6
```

```
octave:8> A=[u1,u2,u3,u4,zeros(3,1)]
A =

   16   22   18   18    0
    2    4    0    2    0
    8    4    4    6    0

octave:11> rref(A)
ans =

     1        0        0      1/2        0
     0        1        0      1/4       -0
     0        0        1      1/4       -0
```

So we have nonzero solutions!

Section 2.3 Linear Independence

Consider the set of vectors $\left\{\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}, \begin{bmatrix} e \\ f \end{bmatrix}\right\}$ where a, b, c, d, e, f can be any rela numbers we like. Do you think it is possible to find values for a, b, c, d, e, f such that these vectors are linearly independent?

Case 1: At least one of the vectors is a multiple of another.

$$x_1\begin{bmatrix} a \\ b \end{bmatrix} + x_2\begin{bmatrix} c \\ d \end{bmatrix} + x_3\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c \\ d \end{bmatrix} = k\begin{bmatrix} a \\ b \end{bmatrix}$$

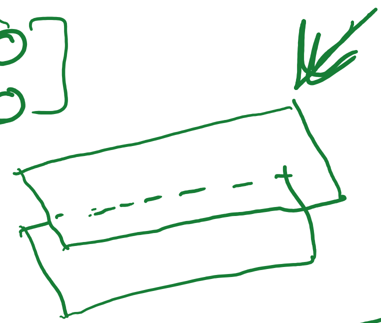
choose $x_1 = -k$ $\quad x_2 = 1 \quad x_3 = 0$

$$(-1)k\begin{bmatrix} a \\ b \end{bmatrix} + 1\begin{bmatrix} c \\ d \end{bmatrix} + 0\begin{bmatrix} e \\ f \end{bmatrix} = 0$$

$$-(1)\begin{bmatrix} c \\ d \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} + 0 = 0 \checkmark$$

Case 2 None of the vectors are multiples of each other.

$$x_1\begin{bmatrix} a \\ b \end{bmatrix} + x_2\begin{bmatrix} c \\ d \end{bmatrix} + x_3\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



$$x_1 a + x_2 c + x_3 e = 0$$

$$k(x_1 b + x_2 d + x_3 f) = (0)k$$

$$a = kb \qquad c = kd \qquad e = kf$$

$$x_1 kb + x_2 kd + x_3 kf = 0$$

$$x_1 b + x_2 d + x_3 f = 0$$

$$x_1\begin{bmatrix} kb \\ b \end{bmatrix} + x_2\begin{bmatrix} kd \\ d \end{bmatrix} + x_3\begin{bmatrix} kf \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Infinately many sol's $x_1, x_2, x_3 \neq 0$

choose $x_3 = 0$

$x_2 = -\frac{b}{d}$

$$x_1 \begin{bmatrix} kb \\ b \end{bmatrix} + x_2 \begin{bmatrix} kd \\ d \end{bmatrix} + x_3 \begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_2 = -\frac{b}{d}$

$x_1 = 1$

$$\begin{bmatrix} kb \\ b \end{bmatrix} - \frac{b}{d} \begin{bmatrix} kd \\ d \end{bmatrix} + 0 = 0$$

$$\begin{bmatrix} kb \\ b \end{bmatrix} - \begin{bmatrix} \frac{b}{\cancel{d}} k\cancel{d} \\ \frac{b}{\cancel{d}} \cancel{d} \end{bmatrix} = 0$$

$$\begin{bmatrix} kb \\ b \end{bmatrix} - \begin{bmatrix} kb \\ b \end{bmatrix} = 0$$

Section 2.3 Linear Independence

$\begin{bmatrix}a\\b\end{bmatrix}\begin{bmatrix}c\\d\end{bmatrix}\begin{bmatrix}e\\f\end{bmatrix}$ $m=3$, $n=2$

> Theorem 2.14: For a set of vectors $\{\vec{u}_1, \vec{u}_2, \cdots \vec{u}_m\}$ in R^n, if $n < m$ they are linearly dependent.

This is because a system of homogeneous equations with more variables than equations must have infinitely many solutions .

Quick Definition: a homogeneous equation is one where the sum of all nonzero terms add up to zero, a homogeneous system is a system of equations made up of homogeneous equations.

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= 0\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= 0\\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= 0\\
\vdots \qquad \vdots \qquad \vdots \qquad\qquad \vdots \quad &\ \ \vdots\\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= 0
\end{aligned}$$

To see why a homogeneous system with less equations than variables must always have infinitely many solutions we only need to think about the possible echelon forms of such a system.

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= 0\\
a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= 0\\
a_{33}x_3 + \cdots + a_{3n}x_n &= 0\\
&\vdots\\
a_{mn-1}x_{n-1} + a_{mn}x_n &= 0
\end{aligned}$$

$x_n = t$

Either we have a free variable coming from a missing pivot position, this is possible too for non-homogeneous systems

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= 0\\
a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= 0\\
a_{33}x_3 + \cdots + a_{3n}x_n &= 0\\
&\vdots\\
0 &= 0
\end{aligned}$$

Or at least one of the left hand sides was a multiple of another, but since all the right hand sides are zero ,its 0=0 giving a free variable. The right hand sides being zero makes it homogeneous and it is that feature that guarantees us a free variable in the case where a left hand side was a multiple of another.

With that in mind, if $n < m$, we have n equations coming from the n components of each vector and m variables coming from each needed multiple of the m vectors. So we have more unknowns than equations in a homogeneous system, therefore there are infinately many solutions one of which is not trivial and the vectors must be linearly dependent.

$$x_1\begin{bmatrix}a_{11}\\ \vdots\\ a_{1n}\end{bmatrix} + x_2\begin{bmatrix}a_{21}\\ \vdots\\ a_{2n}\end{bmatrix} + \cdots + x_m\begin{bmatrix}a_{mn}\\ \vdots\\ a_{mn}\end{bmatrix} = \begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}$$

Section 2.3 Linear Independence

Theorem 2.15: Let $\{\vec{u}_1, \vec{u}_2, \cdots \vec{u}_m\}$ be a set of vectors in R^n. Then this set is linearly dependent if and only if one of the vectors in the set is in the span of the other vectors.

Proof. Suppose the set is linearly dependent

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_m\vec{v}_m = 0 \rightarrow x_1\vec{v}_1 = -x_2\vec{v}_2 - \cdots - x_m\vec{v}_m$$

$$\vec{v}_1 = -\frac{x_2}{x_1}\vec{v}_2 + \cdots - \frac{x_m}{x_1}\vec{v}_m$$

$$\Rightarrow \vec{v}_1 = span\{\vec{v}_2 \cdots \vec{v}_m\}$$

Suppose one vector is in the span of the others

$$\vec{v}_1 = x_2\vec{v}_2 + \cdots + x_m\vec{v}_m$$

$$1\vec{v}_1 - x_2\vec{v}_2 - \cdots - x_m\vec{v}_m = 0$$

Theorem 2.16: Let $\{\vec{u}_1, \vec{u}_2, \cdots \vec{u}_m\} \in R^n$, and suppose we make a matrix with the vectors as columns

$$A = [\vec{u}_1 \;\; \vec{u}_2 \;\; \cdots \;\; \vec{u}_n]$$

When the matrix A is transformed into matrix B with row operations with B in echelon form then

(1) $span\{\vec{u}_1, \vec{u}_2, \cdots \vec{u}_m\} = R^n$ exactly when B has a pivot in every row.
(2) $\{\vec{u}_1, \vec{u}_2, \cdots \vec{u}_m\}$ is linearly independent exactly when B has a piviot position in every column

$$x_1[\;] + x_2[\;] + x_3[\;] = 0$$

(1)

$$\begin{bmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \end{bmatrix}$$

Pivot in every row only

Showed this with theorem 2.8. Quick reasoning is that if there is a pivot position in every row there is no row that can look like $0 = q$ where $q \neq 0$ meaning we won't have a no solution case if we augment the system with any vector from R^n. Thus we could always find a solution to the augmented system with any vector from R^n

(2)

$$\left[\begin{array}{ccc|c} a & b & c & 0 \\ 0 & d & e & 0 \\ 0 & 0 & f & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Pivot in every column only

In this case, because there is a pivot in every column, one row must be all zeros except the m^{th} variable position. This means that when the system is augmented with the zero vector the m^{th} variable must be zero . Since there is also a piviot position in all other columns there can be no free paramaters for the system which means they must all be zero as well. Thus only the trivial solution would exist, if we augmented it with all zeros.

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

Pivot in every Row and column

This case is a combination of the previous two so we get them both! Span and linear independence.

In this case we would say that the original columns of A form a basis for R^n We will talk a lot about the idea of a basis.

Section 2.3 Linear Independence

Theorem 2.18: Let $A = [\vec{u}_1 \;\; \vec{u}_2 \;\; \cdots \;\; \vec{u}_n]$ and $\vec{x} = (x_1, x_2, \cdots, x_m)$. The set $\{\vec{u}_1, \;\; \vec{u}_2, \;\; \cdots \;\; \vec{u}_n\}$ is linearly independent if and only if the homogeneous linear system:

$$A\vec{x} = 0$$

Has only the trivial $(\vec{x} = 0)$ solution.

If we multiply the system out

assume $A\vec{x} = 0$ only $\vec{x} = 0$ as solution

$$[\vec{v}_1 \;\; \vec{v}_2 \cdots \vec{v}_n]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_m v_m = 0$$

$$x_1 = x_2 = \cdots x_m = 0$$

We see it is the same as the definition of linear independence

Theorem 2.19: Let $\vec{x}_p$ be a particular solution of the system

$A\vec{x} = \vec{b}$

And let $\vec{x}_h$ be the homogeneous solution to the system

$A\vec{x} = 0$

Then the vector $\vec{x}_g = \vec{x}_p + \vec{x}_h$ also solves

$A\vec{x} = b$

Just a small note on why this is important. In practice the system $A\vec{x} = b$ may model something physical and the vector $\vec{x}_h$ may not be the zero vector. In that case we want to include it in our solution. So So often when we solve $A\vec{x} = b$ we also must solve $A\vec{x} = 0$. This is especially true in differential equations.

Proof:

$$A\vec{x}_p = \vec{b} \qquad A\vec{x}_n = 0$$

Show

$$A\vec{x}_g = \vec{b} \Rightarrow A(\underbrace{\vec{x}_p + \vec{x}_n}_{g}) = \underbrace{A\vec{x}_p}_{\vec{b}} + \underbrace{A\vec{x}_n}_{0} = \vec{b} + 0$$

$$= \vec{b} \checkmark$$

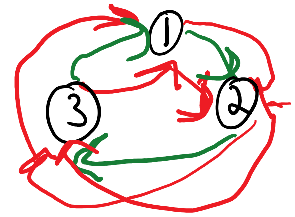
Section 2.3 Linear Independence

A Unifying Theorem. 2.21 Let $S = \{\vec{u}_1, \cdots, \vec{u}_n\}$ be a set of n vectors in R^n, and let $A = [\vec{u}_1 \ \ \vec{u}_2 \ \ \cdots \vec{u}_n]$ Then the following are equivlent.

(1) $S \ spans \ R^n$
(2) $S \ is \ linearly \ independent$
(3) $A\vec{x} = \vec{b} \ has \ a \ unique \ solution \ for \ all \ \vec{b} \ in$ $\mathrm{R^n}$

$$\left[\begin{array}{ccc|c} a & b & c & b_1 \\ 0 & d & e & b_2 \\ 0 & 0 & f & b_3 \end{array}\right]$$

The 123 circle....



$1 \rightleftarrows 2$
$2 \rightleftarrows 3$
$3 \rightleftarrows 1$

For this we need to prove that $(1) \Rightarrow (2)$, $(2) \Rightarrow (3)$, $(3) \Rightarrow (1)$

Theorem 2.16: Let $\{\vec{u}_1, \vec{u}_2, \cdots \vec{u}_m\} \in R^n$, and suppose we make a matrix with the vectors as columns

$$A = [\vec{u}_1 \ \ \vec{u}_2 \ \ \cdots \ \ \vec{u}_n]$$

When the matrix A is transformed into matrix B with row operations with B in echelon form then

(1) $span\{\vec{u}_1, \vec{u}_2, \cdots \vec{u}_m\} = R^n$ exactly when B has a pivot in every row.
(2) $\{\vec{u}_1, \vec{u}_2, \cdots \vec{u}_m\}$ is linearly independent exactly when B has a piviot position in every column

Formal Definition: The span of a set of vectors $\{\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_m\} \in R^n$ denoted $span\{\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_m\}$ is the set of all possible linear combinations

$$c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_m\vec{u}_m$$

of those vectors, where c_i is any real number.

$(1) \Rightarrow (2)$
$If \ span\{S\} = R^n \ by \ theorem$ 2.16 $the \ matrix \ A \ is \ A = [\vec{u}_1 \, \vec{u}_2 \ \ \cdots \ \ \vec{u}_n]$, since there are n rows and n columns, this means there must also be a pivot in every column, thus also by theroem 2.18 the set S is made of linearly independent vectors.

$(2) \Rightarrow (3)$ If the set S is linearly independent again by theorem 2.16 the matrix $A = [\vec{u}_1 \, \vec{u}_2 \ \ \cdots \ \ \vec{u}_n]$ has a piviot position in every column , but there are also only n rows so there is also a piviot position in every row. This means it is impossible to have a row of zeros to the left of the | in any augmented matrix of the form

$$[\vec{u}_1 \, \vec{u}_2 \ \ \cdots \ \ \vec{u}_n | \, \vec{b}]$$

for any arbitray $\vec{b} \in R^n$ we may put to the right of the dividing line. This means a $0 = q$ or $0 = 0$ case is impossible, outlawing any case of no soution or freevariables. Thus one must be able to back solve for a unique solution.

$(3) \Rightarrow (1)$

If we Simply write $A\vec{x} = \vec{b}$ in the in its equivlent vector equation form:

$$x_1\vec{u}_1 + x_2\vec{u}_2 + \cdots + x_n\vec{u}_n = \vec{b}$$

We see that because we suppose there is a solution to $A\vec{x} = \vec{b}$ we know that there exist $x_1 \cdots x_n$ which form a linear combination of the $\vec{u}'s$ to get to $\vec{b}$ for any arbitrary $\vec{b} \in R^n$. Thus, the set S spans R^n by definition of span.

Section 2.3 Linear Independence

Now your turn! Show that all (a)-(d) are equivalent

Theorem 2.20: Let $\{\vec{u}_1, \vec{u}_2, \cdots \vec{u}_m\}, \vec{b}$ be vectors in R^n. Then the following statements are equivelent. This means they if one is true they all are and if one is false they all are.

(a) The set $\{\vec{u}_1, \vec{u}_2, \cdots \vec{u}_m\}$ is linearly independent
(b) The vector equation $x_1\vec{u}_1 + x_2\vec{u}_2 + \cdots + x_m\vec{u}_m = \vec{b}$ has at most one solution for every $\vec{b}$.
(c) The linear system corresponding to $[\vec{u}_1 \;\; \vec{u}_2 \;\; \cdots \vec{u}_m \,|\vec{b}]$ has at most one solution for every $\vec{b}$
(d) The equation $A\vec{x} = \vec{b}$, with $A = [\vec{u}_1 \;\; \vec{u}_2 \;\; \cdots \vec{u}_m]$ has at most one solution for every $\vec{b}$

Hint: $b \Rightarrow c, c \Rightarrow d$ is easy, those are all equivlent statements of the same equation, if one has a solution they all do! So now we just need to argue that $a \Rightarrow b$ and $d \Rightarrow a$

The proof circle

$$d \to a \to b \to c \to d$$

$a \Rightarrow b$ suppose $\vec{v}_1, \dots \vec{v}_n$ is Linearl Indepent
Further suppose that there are Two solutions
$(x_1, x_2, \dots x_m)$ and $(x_1', x_2', \dots x_m')$

Then: $x_1\vec{v}_1 + x_2 v_2 + \dots + x_m\vec{v}_m = \vec{b}$ ①
and $x_1'\vec{v}_1 + x_2' v_2 + \dots + x_m'\vec{v}_m = \vec{b}$ ②

Subtract 2 from 1
$$(x_1 - x_1')\vec{v}_1 + (x_2 - x_2')\vec{v}_2 + \dots + (x_m - x_m')\vec{v}_m = 0$$
but $\vec{v}_1 \dots \vec{v}_m$ are linearly Independent so the only solution to this homogeneous equation is all zero coeff's thus
$$x_1 - x_1' = 0 \quad x_2 - x_2' = 0 \;\cdots\; x_m - x_m' = 0$$
$$\Rightarrow x_1 = x_1' \quad x_2 = x_2' \;\cdots\; x_m = x_m'$$
So there was only one solution all along!

$b \Rightarrow c$ $x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_m\vec{v}_m = b$ and $[v_1 \;\; v_2 \; \dots v_m \,|\, b]$
represent the same system and must have the same solutions.

Section 2.3 Linear Independence

$C \Rightarrow D$

$A\vec{x}=\vec{b}$ and $[v_1 \cdots v_m | b]$ with $A=[v_1 \cdots v_m]$

represent the same system and must have the same solutions

$D \Rightarrow A$

If $A\vec{x}=\vec{b}$ has at most one solution for all $\vec{b}$

then $A\vec{x}=0$ can have at most one solution.

$A\vec{x}=0$ with $A=[v_1 \cdots v_m]$ is the same as

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_m\vec{v}_m = 0$$

Note $x_1=x_2=\cdots=x_m=0$ always solves this homogeneous equation. Since there can be no more than one solution this is the only solution, by definition this means $v_1 \ldots v_m$ are linearly independent.

So now we know $a \Rightarrow b \Rightarrow c \Rightarrow d \Rightarrow a$

so $b \Rightarrow a$ since $b \Rightarrow c \Rightarrow d \Rightarrow a$

and $c \Rightarrow b$ since $c \Rightarrow d \Rightarrow \Rightarrow b$

and $a \Rightarrow d$ since $a \Rightarrow b \Rightarrow c \Rightarrow d$

and $d \Rightarrow c$ since $d \Rightarrow a \Rightarrow b \Rightarrow c$

so we can say $a \Leftrightarrow b \Leftrightarrow c \Leftrightarrow d \Leftrightarrow a$

so they are all equivlent!