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The Cauchy-Goursat Theorem. Cauchy is a name you will hear a lot in the rest of this course as well in many other math courses.

Last class we proved the following theorem:

Theorem: Suppose that a function $f(z)$ is continuous in a domain D. If any one of the following statements are true, then so are all of the others

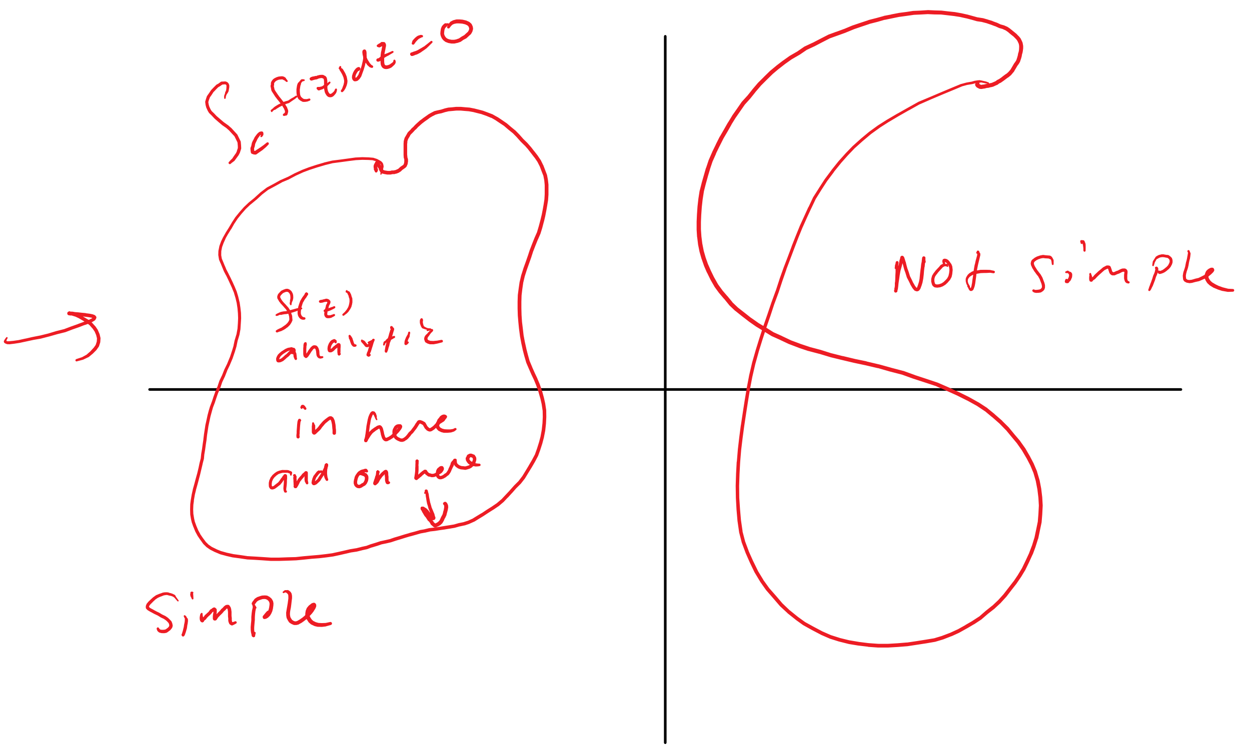
1. $f(z)$ *has an antiderivative* $F(z)$ *Throughout* D
2. *the integrals of* $f(z)$*along contours lying entirely in* D *and extending from any fixed point* z_1 *to any fixed point* z_2 *all have the same value, that is* $\int_C f(z)dz = \int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1)$
3. *the integrals of* $f(z)$ *around closed contours lying entirely in* D *are all* 0.

With this theorem we showed that if one of the 3 statements were true all of them were. Now we will present another theorem which gives a specific condition on $f(z)$, namely analycity on and interior to the contour, which if satisfied garentees that the integral around a simple closed contour is zero.

Cauchy-Goursat Theorem:

If a function $f(z)$ is analytic at all points interior to and on a simple closed curve countour C, then:

$$\int_C f(z)dz = 0$$



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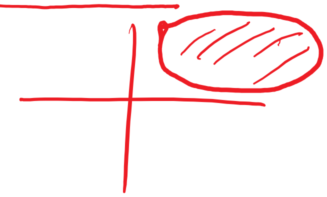
Let's prove it with one extra assumption. We assume $f(z) = u + iv$ is analytic and thus continuious, but we will also assume that $f'(z)$ is continuious as well. Note the C-G theorem does not require the continuity of $f'(z)$ but the proof with out that is a lecture in itself. Cauchy proved it with our extra assumption, Goursat was able to show that we need not assume that $f'(z)$ is continuous. However, as it turns out, it is anyway. As we will see later the derivativive of an analytic function is it's self analytic and thus continuious it is the removal of the assumption of continuity of f' that allows us to prove it! Enough rambling.. On to the proof.

For this proof we will need to recall Green's Theorem from Calc III namely that

For two continuous real valued functions $P(x,y)$ and $Q(x,y)$ that have continuious first order partials in some closed region R we have the following equality:

$\frac{\partial Q}{\partial x}$ $\quad \partial P/\partial y$

$$\int_C Pdx + Qdy = \int_A (Q_x - P_y)dA$$



R $\mathbb{R}^2$

As you might guess for our analytic function $f(z) = u(x,y) + iv(x,y)$, u and v will be our P and Q

We know that the contour integral

$z(t) \qquad a \le t \le b$

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt$$

Let $f(z) = u(x,y) + iv(x,y)$ and $z(t) = x(t) + iy(t)$ then $z'(t) = x'(t) + iy'(t)$, and think of $x'(t) = \frac{dx}{dt}$, $y'(t) = \frac{dy}{dt}$

$\frac{dx}{dt}dt = dx$

Let's plug this in to $\int_a^b f(z(t))z'(t)dt$

$f(z(t)) = u(x(t), y(t)) + i\, v(x(t), y(t))$

$$\int_a^b (u(x,y) + iv(x,y))(x'(t) + iy'(t))\,dt$$

$$\int_a^b ux' + ivx' + iuy' - vy'\,dt$$

$$\int_a^b ux' - vy' + i(vx' + uy')\,dt$$

$$\int_a^b (ux' - vy')dt + i\int (vx' + uy')dt$$

$$\int_a^b \left(u\frac{dx}{dt} - v\frac{dy}{dt}\right)dt + i\int_a^b \left(v\frac{dx}{dt} + u\frac{dy}{dt}\right)dt$$

$$= \boxed{\int_a^b udx - vdy + i\int vdx + udy}$$

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so

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt = \int_C udx - vdy + i\int_C vdx + udy$$

If we apply greens theorem $\int_C Pdx + Qdy = \int_A (Q_x - P_y)dA$ to each integral on the left:

$$\int_C \underbrace{u}_{P}dx + \underbrace{(-v)}_{Q}dy + i\int_C \underbrace{v}_{P}dx + \underbrace{u}_{Q}dy$$

$$\int_R -v_x - u_y\,dA + i\int_R u_x - v_y\,dA$$

Then:

$$\int_C f(z)dz = \int_R (-v_x - u_y)dA + i\int_R u_x - v_y dA$$

Now we know that $f(z)$ was analytic so the C-R equations apply! $u_x = v_y$, $u_y = -v_x$

$$\int_R \underbrace{(-v_x - u_y)}_{0}dA + i\int_R \underbrace{u_x - v_y}_{0}dx$$

$u_x = v_y$

$u_x - v_y = 0$

$u_y = -v_x$

$0 = -v_x - u_y$

$$\int_C f(z)dz = \int 0 + i\int 0 = 0$$

$\boxed{\text{And we have that } \int_C f(z)dz = 0}$

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Cauchy-Goursat Theorem:

If a function $f(z)$ is analytic at all points interior to and on a simple closed curve countou C, then:

$$\int_C f(z)dz = 0$$

Homework Example:

1. Apply the Cauchy-Goursat theorem to show that

$$\int_C f(z)dz = 0$$

for the contour C being the unit circle about the origin, or determine that Cauchy-Goursat theorem does not apply.

(a) $f(z) = \frac{z^2}{z-3}$

(b) $f(z) = ze^{-z}$

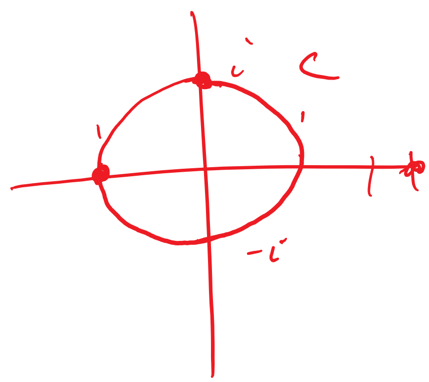
(c) $f(z) = \frac{z}{2z-i}$

(d) $f(z) = \tan(z) = \frac{\sin(z)}{\cos(z)}$

(e) $f(z) = Log(z+2)$

(f) $f(z) = \frac{1}{z^2+3z+2}$

(g) $f(z) = log(z)$, any branch.



d) $f(z) = \tan(z)$ $\tan(z)$ is not analytic on the unit circle at $\theta = \pi/2$

$\Rightarrow$ C-G does not apply

f) $f(z) = \frac{1}{z^2+3z+2} = \frac{1}{(z+2)(z+1)}$ Not analytic at $z=-2$, $z=-1$

C-G does not apply

a) $\frac{z^2}{z-3} = f(z)$ $f(z)$ Not Analytic at $z=3$ Not on C or in R

solves $\int_C f(z)dz = 0$

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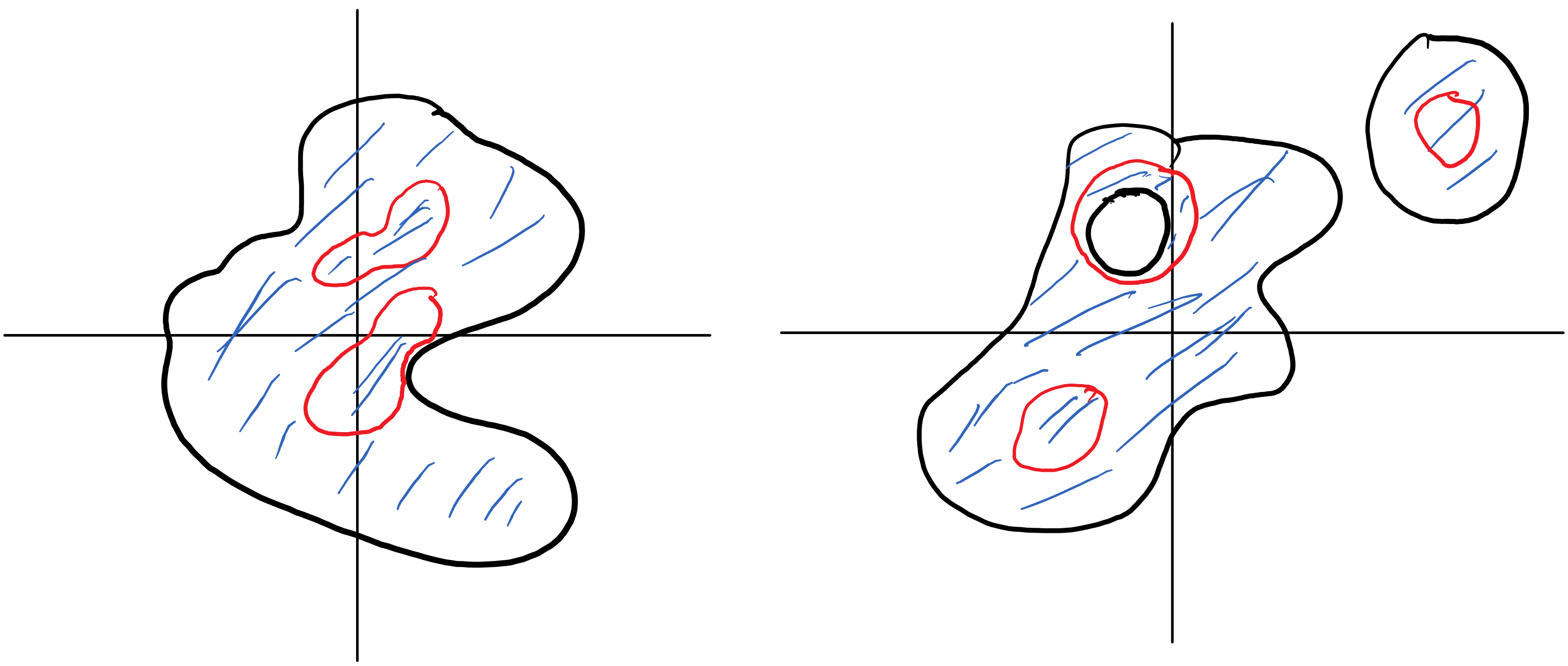
Simply Connected Domains

A simply connected domain D is a domain such that every simple closed contour with in D encloses only points of D.



Simply Connected

Not Simply Connected

It turns out that if a function $f(z)$ is analytic in a simply connected domain then we can extend the C-G theorem to contours which themselves are not simple.

Theorem: if a function f is analytic in a simply connected domain D then:

$$\int_C f(z)dz = 0$$

for **every** closed contour in lying in D

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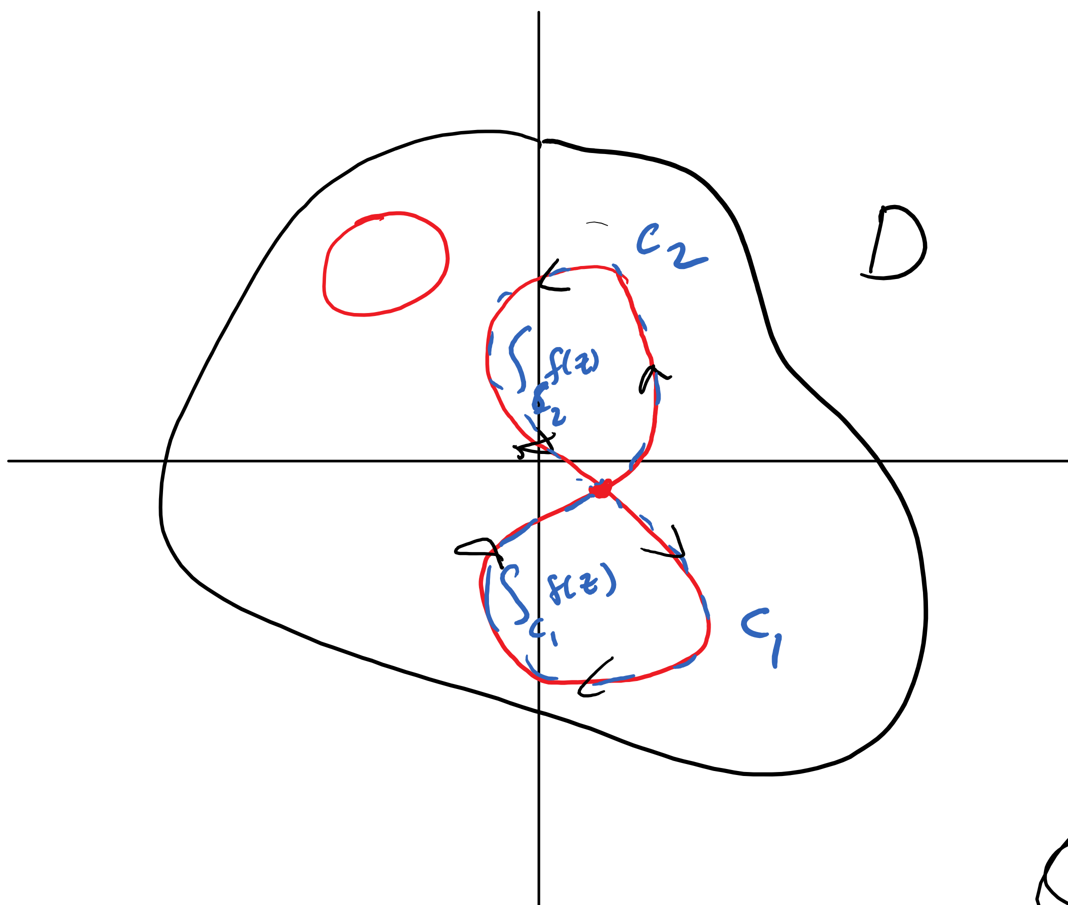
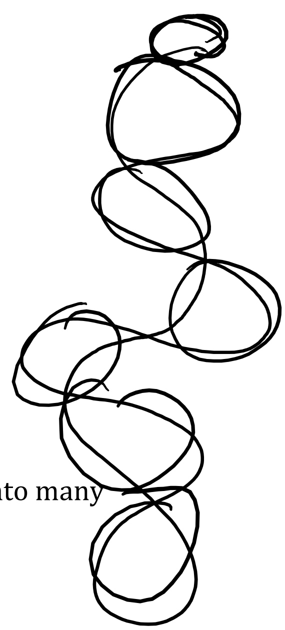
Theorem: if a function f is analytic in a simply connected domain D then:

$$\int_C f(z)dz = 0$$

for **every** closed contour in lying in D

This is easy to see if C is already simple as C-G directly applies. Let's consider the case where C intersects itself a finite number of times.





Since $\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$ we can split up our non simple closed contour into many simple ones and just apply C-G to each!

The case of infinite intersections is not so obvious though....

$$\int_{C_1} \cancelto{0}{f(z)dz} + \int_{C_2} \cancelto{0}{f(z)dz} = \int_C f(z)dz = 0$$

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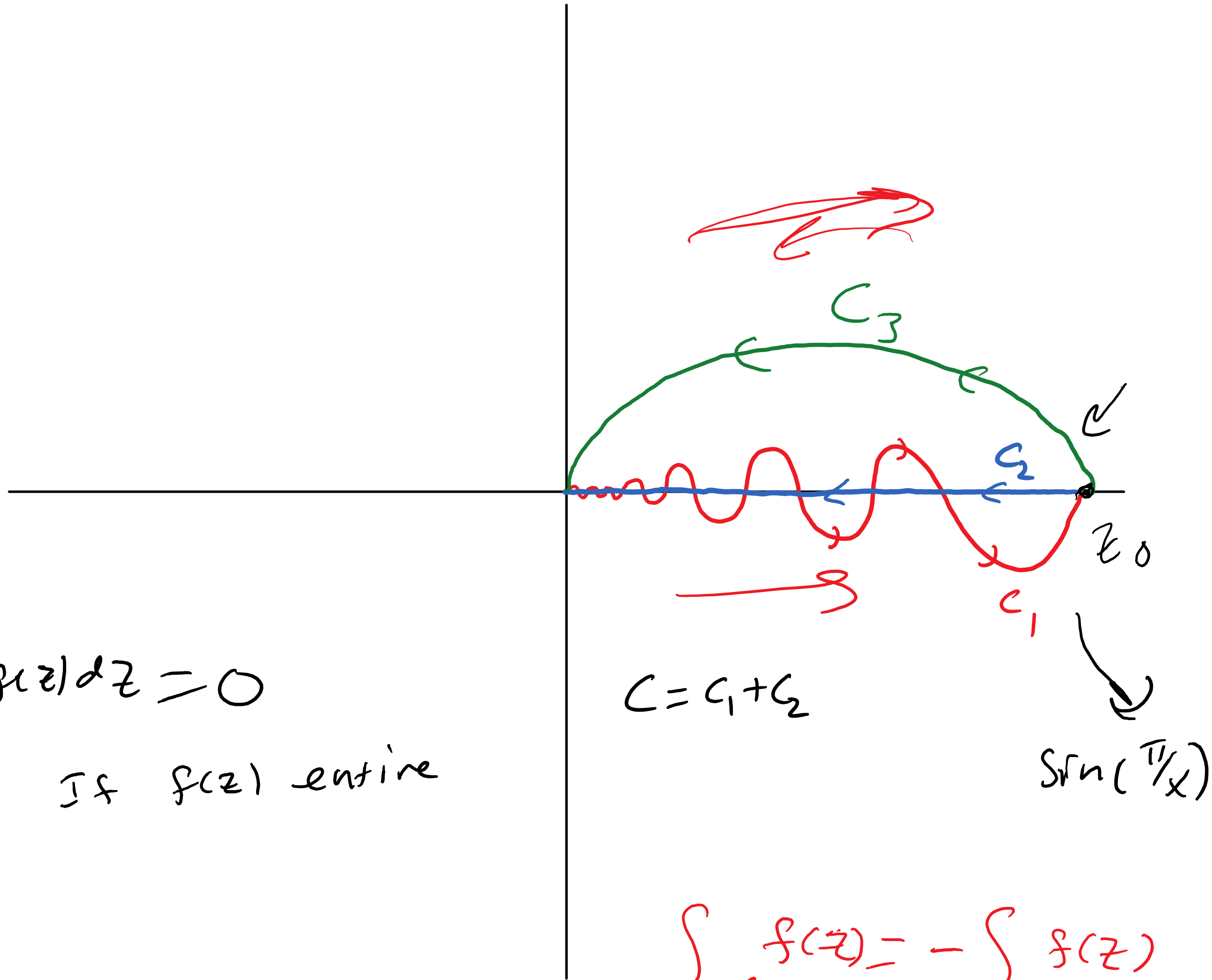
An example..



$$\int_C f(z)\,dz = 0$$

If $f(z)$ entire

$$C = C_1 + C_2$$

$$\int_{-C} f(z) = -\int_C f(z)$$

The trick to show that $\int_C f(z)dz = 0$ for this crazy contour will be to employ path independence and a third simple contour C_3 which links z_0 back to the orign.

$$\underline{\int_{C_2} f(z)\,dz = \int_{C_3} f(z)\,dz}\ , \quad \underline{\int_{C_1} f(z)\,dz = \int_{-C_3} f(z)\,dz}$$

$$\int_C f(z)\,dz = \int_{C_1} f(z) + \int_{C_2} f(z) = \int_{-C_3} f(z)\,dz + \int_{C_3} f(z)\,dz$$

$$-\int_{C_3} f(z)\,dz + \int_{C_3} f(z)\,dz = 0$$

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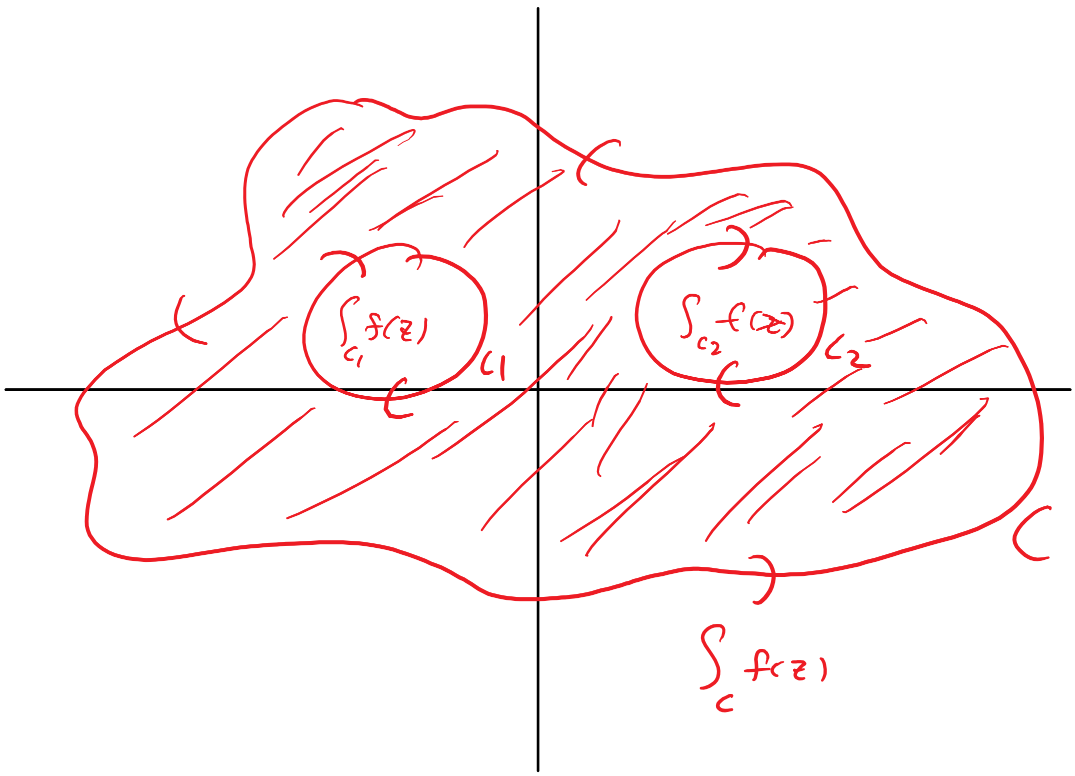
Multiply Connected Domains

A domain that is not simply connected is said to be **multiply connected**



$$\int_C f(z) + \int_{C_1} f(z) + \int_{C_2} f(z) = 0$$

We can again extend the C-G theorem a bit in these domains with the following theorem.

Suppose that C is a simple closed contour positively oriented

1. C is a simple closed contour which is positively oriented
2. C_k $(k = 1,2, \ldots n)$ are simpleclosed contours interior to C, all negatively oriented(clockwise) and disjoint with no comon interior points.

Then if $f(z)$ is analytic on all of these contours and throughout the multiply connected domain consisting of the points inside C and exterior to each C_k then

$$\int_C f(z)dz + \sum_{k=1}^{n} \int_{C_k} f(z)dz = 0$$

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We can again extend the C-G theorem a bit in these domains with the following theorem.

Suppose that C is a simple closed contour positively oriented

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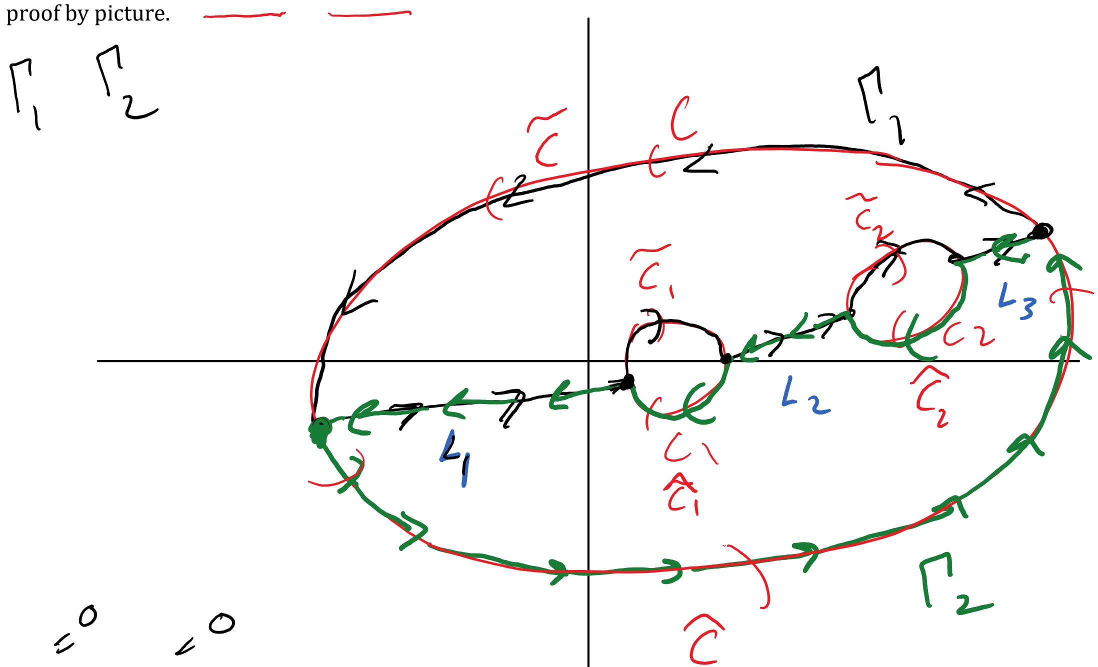
Then if $f(z)$ is analytic on all of these contours and throughout the multiply connected domain consisting of the points inside C and exterior to each C_k then

$$\int_C f(z)dz + \sum_{k=1}^{n} \int_{C_k} f(z)dz = 0$$

$$\int_C f(z)dz + \int_{-C} f(z)dz = 0$$

Keeping in mind that $\int_c f(z)dz = -\int_{-c} f(z)dz$ for any contour and the C-G theorem, we attempt a proof by picture.



$$\overset{=0}{\boxed{\int_{\Gamma_1} f(z)}} + \overset{=0}{\boxed{\int_{\Gamma_2} f(z)}} = \int_{\tilde{C}} f + \cancel{\int_{L_1} f} + \int_{\tilde{C}_1} f + \cancel{\int_{L_2} f} + \int_{\tilde{C}_2} f + \cancel{\int_{L_3} f} + \int_{\hat{C}} f + \cancel{\int_{-L_3} f} + \int_{\hat{C}_2} f + \cancel{\int_{-L_2} f} + \int_{\hat{C}_1} f + \cancel{\int_{-L_1} f}$$

What we just did was connect the contours using polygonal contours in such a way as to make two simple contours Γ_1 *and* Γ_2 to which we could apply the C-G theorem. The connections were done in such a way that we traverse them twice to cancel out their contrubution to the contour inegral.

$$\int_{\tilde{C}} f + \int_{\hat{C}} f = \int_C f$$

$$\int_{\tilde{C}_1} f + \int_{\hat{C}_1} f = \int_{C_1} f$$

$$\int_{\Gamma_1} f(z) + \int_{\Gamma_2} f(z) = 0 = \int_C f + \int_{C_1} f + \int_{C_2} f$$

$$c_1 \qquad c_1 \qquad c_1$$

$$\int_{\tilde{C}_2} f + \int_{\hat{C}_2} f = \int_{C_2} f$$

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An important Corollary of the previous theorem is what we call the **principle of deformation**

Let C_1, C_2 be two positively oriented simple closed contours with C_1 lying in the interior of C_2. If a function $f(z)$ is analytic on the closed region consisting of the two contours and all points between them then:

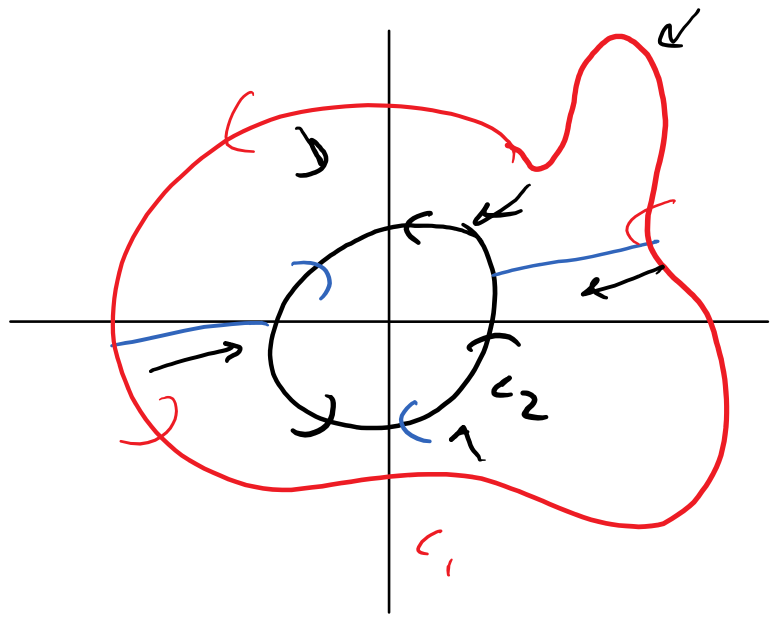
$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

To see why we go the negative direction on C_1 and apply the previous theorem. That is:

$$\int_{C_2} f(z)dz + \int_{-C_1} f(z)dz = 0$$

$$\int_{C_2} f(z)dz = -\int_{-C_1} f(z)dz$$

$$\int_{C_2} f(z)dz = \int_{C_1} f(z)dz$$

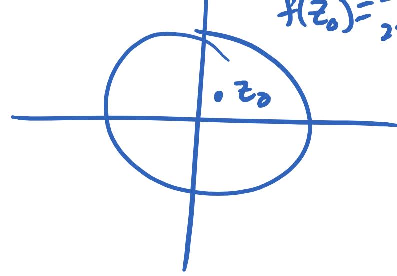


This theorem leads us to a very important formula… The **Cauchy Integral Formula.**

Let $f(z)$ be analytic everywhere inside and on a simple closed contour C which is positively oriented. If z_0 is any point in the interior of C then

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}dz$$

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}$$



Or another form is

$$\int_C \frac{f(z)}{z - z_0}dz = 2\pi i f(z_0)$$

Note that in this form, the function being integrated $\frac{f(z)}{z-z_0}$ has a singularity at z_0 which determines the value of the integral… Neat!

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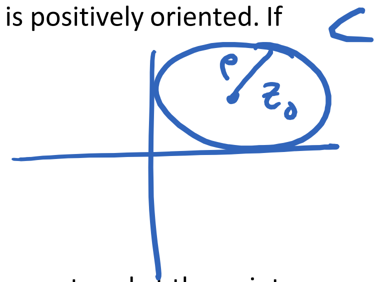
The **Cauchy Integral Formula.**

Let $f(z)$ be analytic everywhere inside and on a simple closed contour C which is positively oriented. If z_0 is any point in the interior of C then



$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}\,dz$$

$\int_{C_\rho} f(z)\,dz$

To prove this let's first calculate $\int_{c_\rho} \frac{1}{z - z_0}\,dz = 2\pi i$ Where c_ρ is a circle of radius ρ centered at the point z_0

$$z(\theta) = \rho e^{i\theta} + z_0 \qquad z'(\theta) = \rho i e^{i\theta} \qquad 0 \le \theta \le 2\pi$$

$$f(z) = \frac{1}{z - z_0}$$

$$\int_a^b f(z(\theta))\, z'(\theta)\, d\theta =$$

$$\int_0^{2\pi} \frac{1}{\rho e^{i\theta} + \cancel{z_0} - \cancel{z_0}}\, \rho i e^{i\theta}\, d\theta = \int_0^{2\pi} \frac{\cancel{\rho} i \cancel{e^{i\theta}}}{\cancel{\rho} \cancel{e^{i\theta}}}\, d\theta$$

$$\int_0^{2\pi} i\, d\theta = \theta i \Big|_0^{2\pi} = 2\pi i - 0 i = \boxed{2\pi i}$$

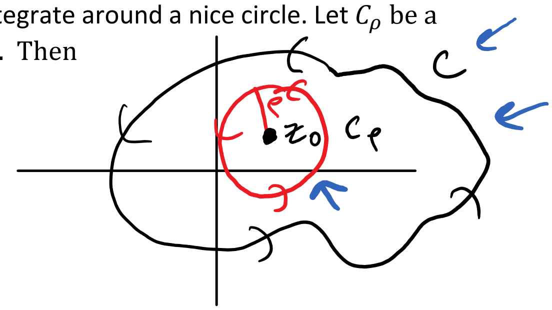
This theorem leads us to a very important formula... The **Cauchy Integral Formula.**

Let $f(z)$ be analytic everywhere inside and on a simple closed contour C which is positively oriented. If z_0 is any point in the interior of C then

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz$$

Proof:

First we will use our deformation principle so that we can integrate around a nice circle. Let C_ρ be a circle of radius ρ centered at z_0 lying inside the contour C. Then



$$\frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz = \frac{1}{2\pi i}\int_{C_\rho} \frac{f(z)}{z-z_0}dz$$

We want to show that $f(z_0) = \frac{1}{2\pi i}\int_{C_\rho}\frac{f(z)}{z-z_0}dz$ this is equivlent to showing $\int_{C_\rho}\frac{f(z)}{z-z_0}dz - 2\pi i f(z_0) = 0$

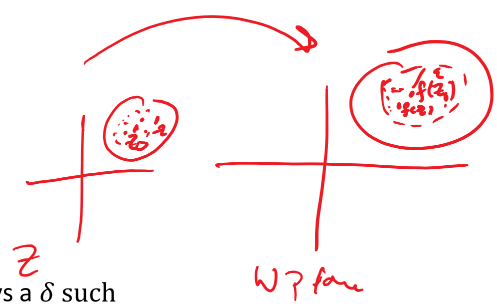
$2\pi i f(z_0) = \int_{C_\rho}\frac{f(z)}{z-z_0}$

$=2\pi i$

From what we just calculated, $\int_{C_\rho}\frac{dz}{z-z_0} = 2\pi i$ thus $2\pi i f(z_0) = f(z_0)\int_{C_\rho}\frac{dz}{z-z_0}$ If we substutute this into the equation above we can see that.

$\int_{C_\rho}\frac{f(z)}{z-z_0}dz - \int_{C_\rho}\frac{f(z_0)}{z-z_0}dz$

$$\int_{C_\rho}\frac{f(z)}{z-z_0}dz - 2\pi i f(z_0) = \int_{C_\rho}\frac{f(z)-f(z_0)}{z-z_0}dz$$



Now recall that $f(z)$ is analytic and therefore it is continuious, this means that there is always a δ such that for any $\epsilon > 0$ $|f(z)-f(z_0)| \le \epsilon$ Whenever $|z-z_0| \le \delta$

So then let $\epsilon > 0$ I am guarenteed a δ I can also choose the redius of my circle, I choose $\rho = \delta$ then since $|z-z_0| < \delta = \rho$ I can say that $|z-z_0| < \rho$ and $|f(z)-f(z_0)| \le \epsilon$ Now we estimate our integral above

$$\left|\int_{C_\rho}\frac{f(z)-f(z_0)}{z-z_0}dz\right| \le \int_{C_\rho}\left|\frac{f(z)-f(z_0)}{z-z_0}\right|dz \le \int_{C_\rho}\frac{|f(z)-f(z_0)|}{|z-z_0|}dz \le \int_{C_\rho}\frac{\epsilon}{\rho}dz = \frac{\epsilon}{\rho}\boxed{\int_{C_\rho}dz} = \frac{\epsilon}{\cancel{\rho}}2\pi\cancel{\rho} = 2\pi\epsilon$$

$<\delta=\rho$

Phew! Now remember that epsilon was any positive number no matter how close to zero, so the left most integral must be zero or there would be some epsilon close enough to zero for which it would be larger.

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So we can now say:

$$\int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0} dz = \int_{C_\rho} \frac{f(z)}{z - z_0} dz - 2\pi i f(z_0) = 0$$

Or that

$$\int_{C_\rho} \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0)$$

And since

$$\int_{C_\rho} \frac{f(z)}{z - z_0} dz = \int_{C} \frac{f(z)}{z - z_0} dz$$

We finally have that

$$\int_{C} \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0)$$

Or equivalently

$$\boxed{f(z_0) = \frac{1}{2\pi i} \int_{C} \frac{f(z)}{z - z_0} dz}$$

Note that this says something interesting, it tells us that if f is to be analytic within and on a simple closed contour C then the vales of $f(z)$ interior to C are determined by the values of $f(z)$ on C.

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Homework Example

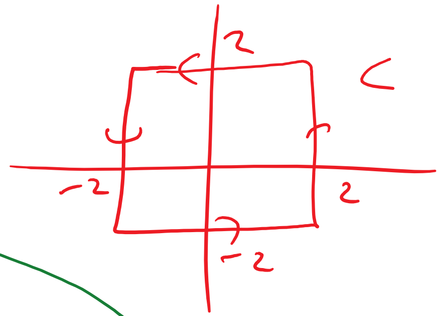
2. Let C denote the positively oriented boundary of the square whose sides lie along $x \pm 2$ and $y \pm 2$. Evaluate the following integrals.

   (a) $\int_C \frac{e^{-z}}{z-\frac{\pi i}{2}}dz$

   (b) $\int_C \frac{z}{2z+1}dz \rightarrow \frac{z}{2z+1} = \frac{1}{2}\left(\frac{z}{z-(-1/2)}\right)$

   (c) $\int_C \frac{\cosh(z)}{z}$

$z-0$



analytic!

a) $\int_C \frac{e^{-z}}{z-\pi i/2}dz$

$$\int_C \frac{f(z)}{z-z_0}dz = 2\pi i f(z_0)$$

$$= 2\pi i e^{-i\pi/2}$$

$$= 2\pi i(-i) = \boxed{2\pi}$$