Lattice Boltzmann-Based Immersed Boundary Calculations

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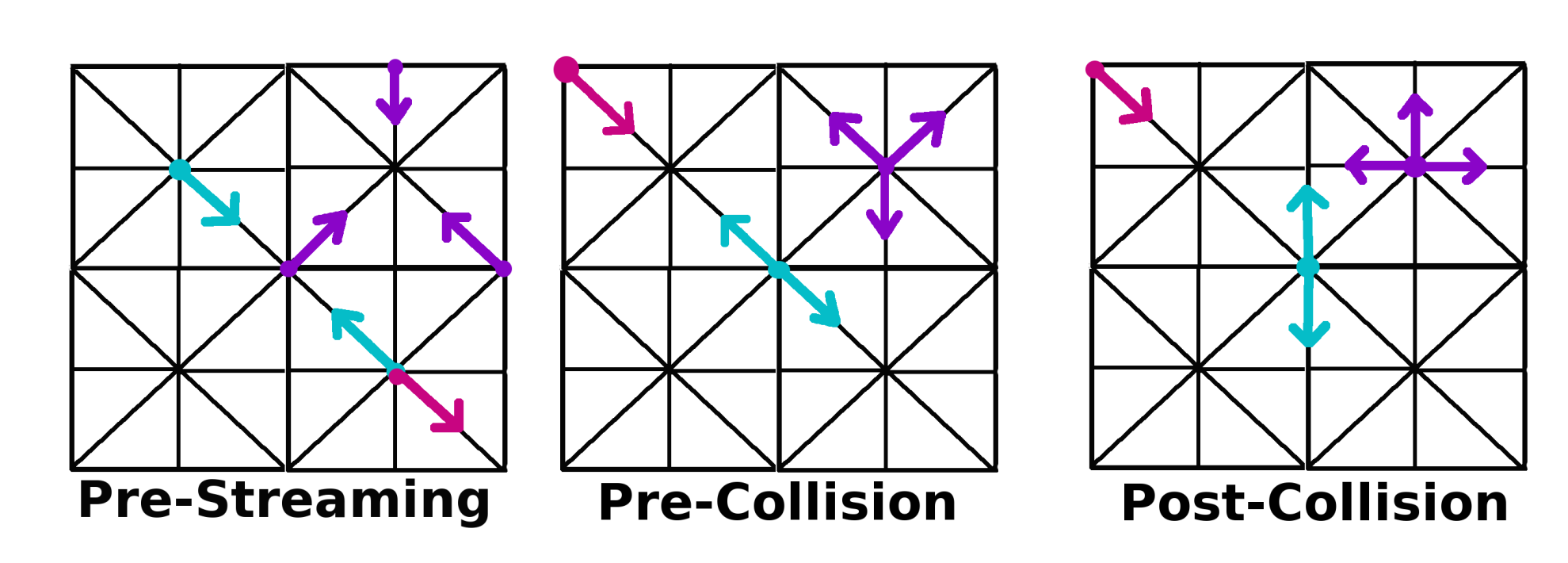
Abstract

The lattice Boltzmann Method (LBM) is a relatively new computational approach to solving fluid flow problems that uses a mesoscopic particle-based technique. Its advantage over traditional computational fluid dynamics is that it bypasses the need to solve for the pressure gradient and so all calculations become local ones. This poster presents a lattice Boltzmann algorithm that incorporates the Immersed Boundary Method (IBM). This method is being developed with the goal of simulating fluid filled deformable particles, such as platelets or red blood cells. The accuracy and speed of this new method will be compared to the original Navier Stokes Immersed Boundary Method.

Lattice Boltzmann: Origin

- The LBM originates from discrete kinetic theory lattice-gas-automata where gas particles can
  - move to neighboring nodes
  - collide with each other.
- Instead of keeping track of discrete particles, the LBM keeps track of particle distribution functions $f(\mathbf{x}, t)$ at each node. This eliminates noise from the model but keeps the simple evolution equations.



- The LBM can also be seen as a discretization of the continuous Boltzmann equation [3] if we assume a simplification of the collision operator to a single relaxation time towards equilibrium:

$$\frac{\partial f(\mathbf{x}, \mathbf{e}, t)}{\partial t} + \mathbf{e} \cdot \nabla f(\mathbf{x}, \mathbf{e}, t) = C(f) = \tfrac{1}{\lambda}(f(\mathbf{x}, \mathbf{e}, t) - g(\mathbf{x}, \mathbf{e}, t)),$$

where λ is the relaxation time and $g(\mathbf{x}, \mathbf{e}, t)$ is the Maxwell Boltzmann distribution about the macroscopic velocity of the fluid, $\mathbf{u}$.

Lattice Boltzmann: Governing Equations

- During a given time step, the particle distributions f_i advect to corresponding neighboring nodes (in the direction of $\mathbf{e}_i$) and relax toward equilibrium:

$$\underbrace{f_i(\mathbf{x} + \delta_t \mathbf{e}_i, t + \delta_t) - f_i(\mathbf{x}, t)}_{\text{Advection}} = \underbrace{\frac{1}{\tau}(f_i(\mathbf{x}, t) - f_i^{eq}(\mathbf{x}, t))}_{\text{Relaxation/Collision}},$$

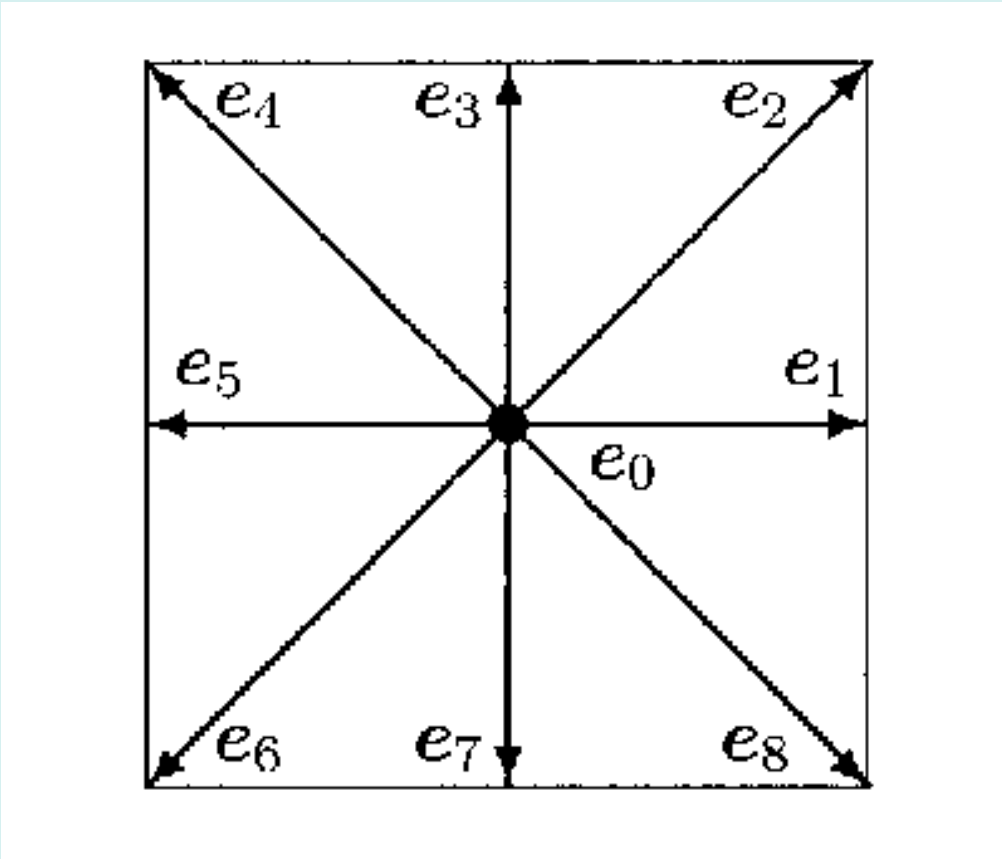
where $\tau = \frac{\delta_t}{\lambda}$ and i's denote the velocity direction at any particular node.

- The equilibrium distribution is an expansion in $\mathbf{u}$ of the Maxwell Boltzmann distribution function up to second order:

$$f_i^{eq}(\rho, \mathbf{u}) = \rho w_i \left(1 + \frac{\mathbf{e}_i \cdot \mathbf{u}}{c^2} + \frac{(\mathbf{e}_i \cdot \mathbf{u})^2}{2c^4} - \frac{\mathbf{u} \cdot \mathbf{u}}{2c^2}\right).$$

- We typically discretize velocity into eight non-zero directions since we want a Cartesian geometry and the more simple square velocity model has non-physical macroscopic properties. We choose the appropriate weighting by distance for each velocity:

$$\begin{cases} w_0 = 4/9 & \\ w_i = 1/9 & i = 1, 3, 5, 7 \\ w_{2i} = 1/36 & i = 2, 4, 6, 8. \end{cases}$$

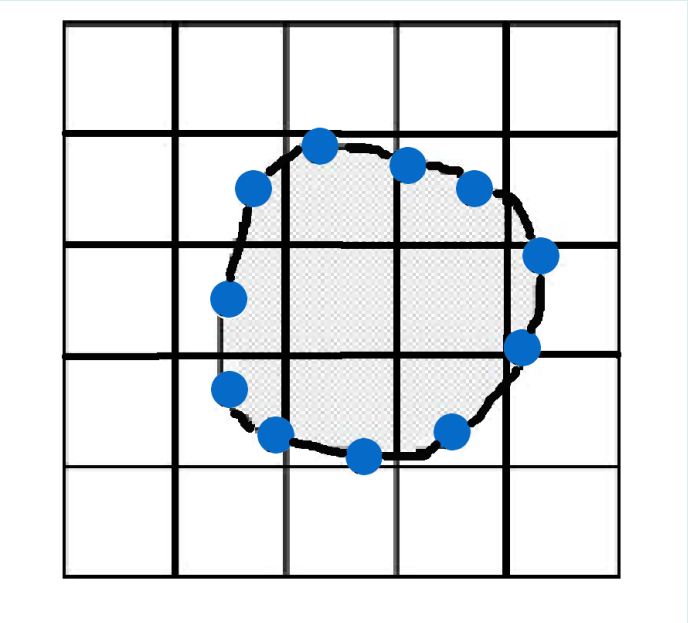


- The governing equations conserve macroscopic density and momentum:

$$\begin{aligned} \sum_i f_i(\mathbf{x}, t) = \sum_i f_i^{eq}(\mathbf{x}, t) &= \rho(x, t) \\ \sum_i f_i(\mathbf{x}, t)\mathbf{e}_i = \sum_i f_i^{eq}(\mathbf{x}, t)\mathbf{e}_i &= \rho(\mathbf{x}, t)\mathbf{u}(\mathbf{x}, t). \end{aligned}$$

- We recover these macroscopic properties of the fluid by summing over i at each node at each time step.

Immersed Boundary Method



- We want to include a fluid-structure interaction, the goal of which is being able to model biological fluid dynamics.
- We assume a thin massless membrane that resists stretching. This membrane lives in a Lagrangian frame and is coupled to an Eulerian fluid velocity field.
- $\mathbf{X}(q, t)$ denotes the location of a point q at time t in the Lagrangian frame.
- Simple elastic force rule:

$$\mathbf{F}(q, t) = s\frac{\partial^2 \mathbf{X}(q, t)}{\partial q^2}$$

- Coupling (boundary moves with fluid velocity)

$$\frac{\partial \mathbf{X}(q, t)}{\partial t} = \mathbf{u}\left(\mathbf{X}(q, t), t\right)$$

- Coupling (boundary force affects fluid for Incompressible Navier Stokes)

$$\rho\left(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}\right) = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{F}$$

Force Term

- In order to couple the LBM to an Immersed Boundary, we need to add a time dependent, spatially varying force term into the lattice Boltzmann equations.
- Since the Navier Stokes equations can be recovered from a Chapman-Enskog expansion [1] of the governing equations, we alter our equations in order to get the correct macroscopic approximation to the Navier Stokes equations with an external force [2].

$$\begin{aligned} f_i(\mathbf{x} + \mathbf{e}_i \delta t, t + \delta_t) - f_i(\mathbf{x}, t) &= -\frac{1}{\tau}\left[f_i(\mathbf{x}, t) - f_i^{eq}\left(\mathbf{u} + \frac{\delta_t \mathbf{F}}{2\rho}\right)\right] \ldots \\ &\quad + w_i \delta_t \left[\frac{\alpha \mathbf{F} \cdot \mathbf{e}_i}{c_s^2} + \frac{\mathbf{C} : (\mathbf{e}\mathbf{e}^T - c_s^2 \mathbf{I})}{2c_s^4}\right]. \end{aligned}$$

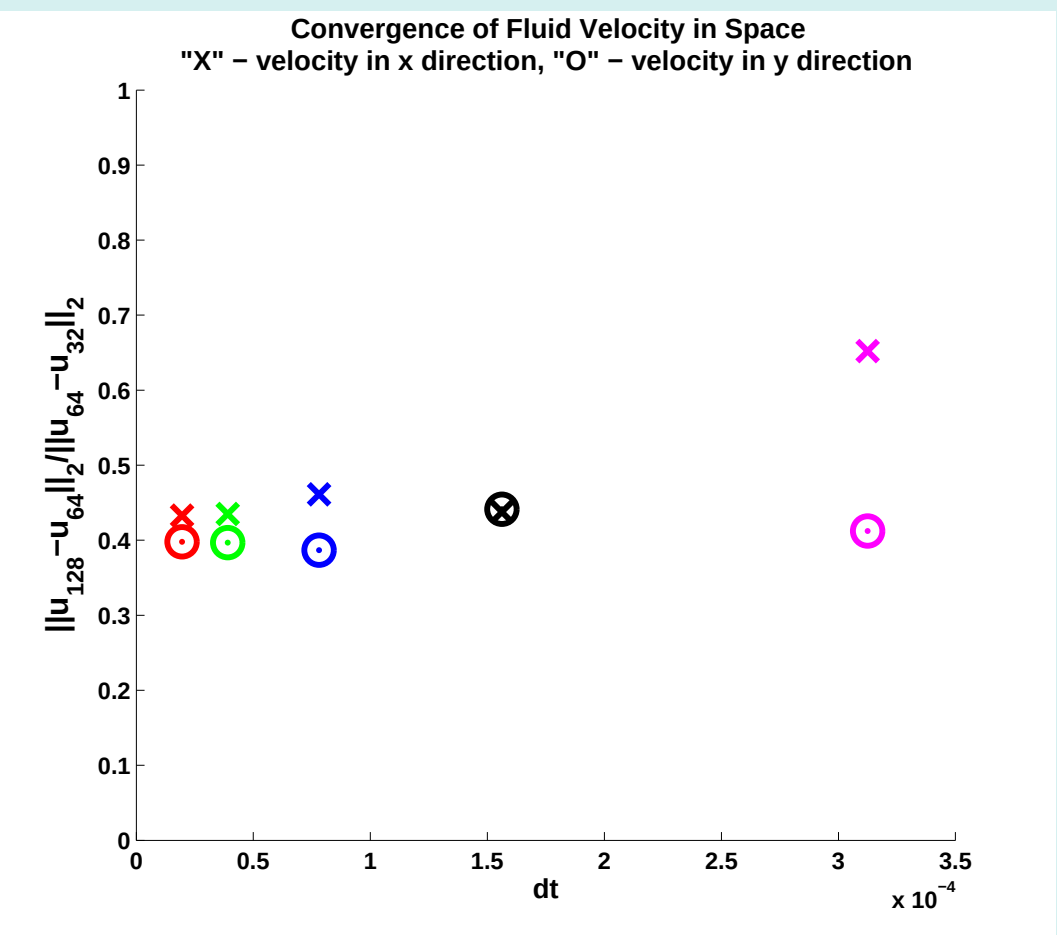
- We need to redefine the momentum density $\mathbf{v} = \mathbf{u} + \frac{\delta_t \mathbf{F}}{2\rho}$ to include the Immersed Boundary force, since it has large spatial variations [4].
- When we recover the Navier Stokes equations from the lattice Boltzmann equations, we find that:
  - the macroscopic viscosity is $\mu = \rho c_s^2 \left(\tau - \frac{1}{2}\right) \delta_t$
  - the pressure is given by an equation of state: $p = \rho c_s^2$.

Lattice Boltzmann-Immersed Boundary Algorithm

1. Calculate force on IB ($F_{ib(\mathbf{x},t)}$).
2. Spread force onto Eulerian grid ($F_{LB}(\mathbf{x}, t)$).
3. Advect f_i's to neighboring nodes.
4. Sum over i at each node to determine $\rho(\mathbf{x}, t)$ and $\mathbf{u}(\mathbf{x}, t)$.
5. Calculate $f_i^{eq}(\rho, \mathbf{u})$.
6. Collide (relax to equilibrium) by $1/\tau$.
7. Move IB with fluid velocity.

Convergence in Space

- For smooth external forces, convergence is second order in space and time, although we do require a small enough time step to ensure incompressibility.
- We test convergence for the ellipse problem with periodic boundary conditions where $\mu = 0.01$, $\rho = 1$, $Re \approx 10^{-2}$, and $s = 1$. Initially the ellipse is set to $r_x = 0.40$, $r_y = 0.15$.

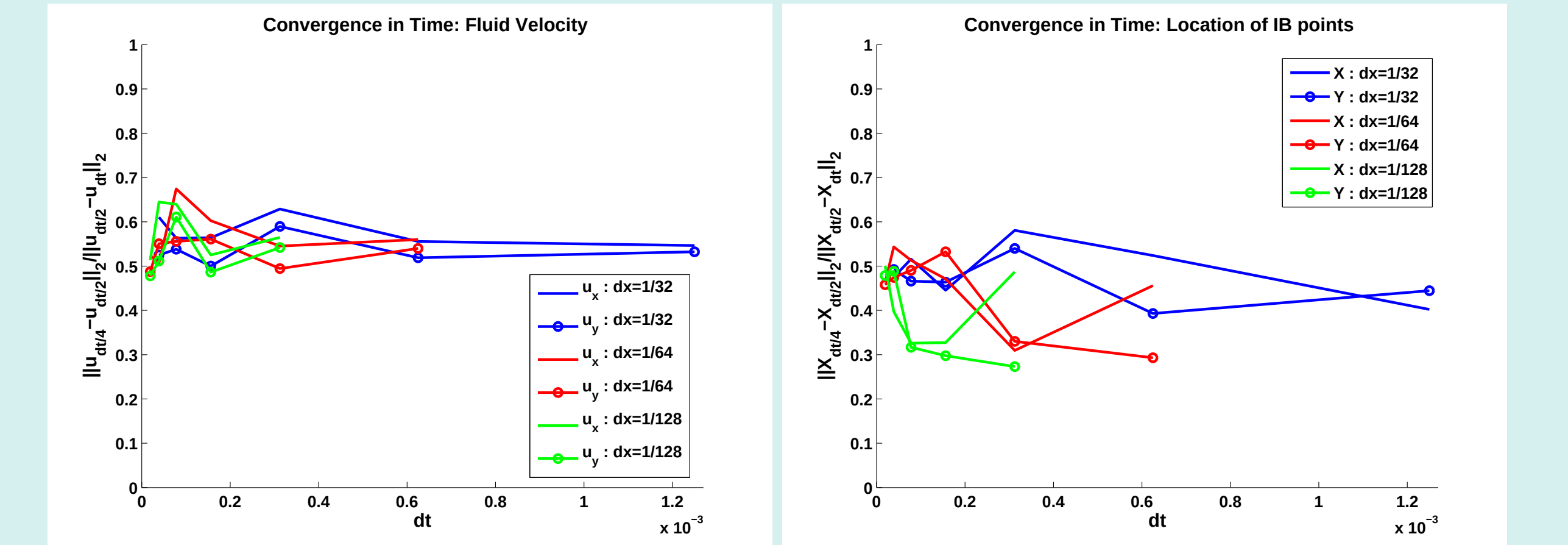


- We check the convergence rate by looking at size of the fraction

$$\frac{\|u_{dx/4} - u_{dx/2}\|_2}{\|u_{dx/2} - u_{dx}\|_2}.$$

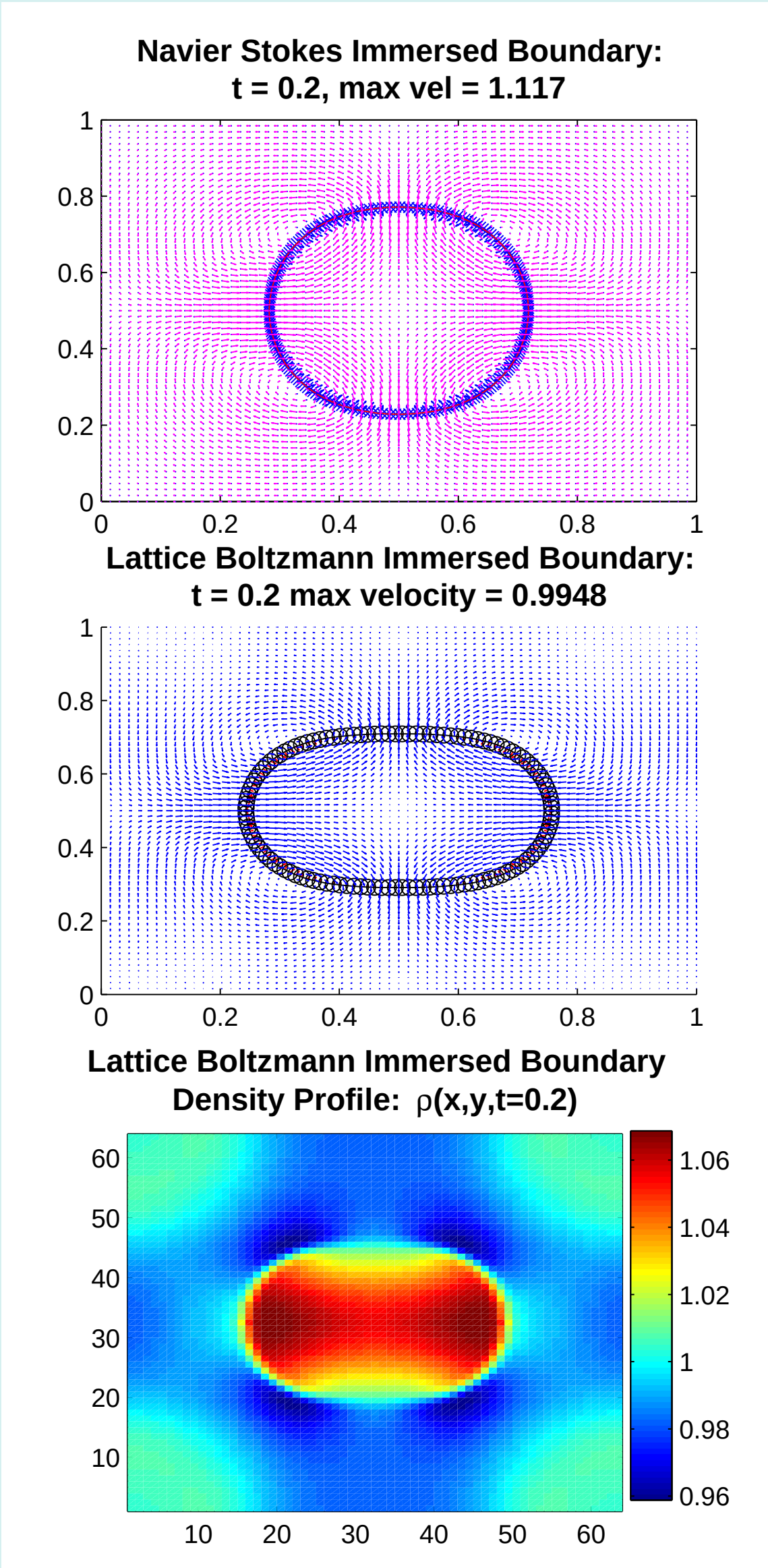
for the fluid velocity.

Convergence in Time

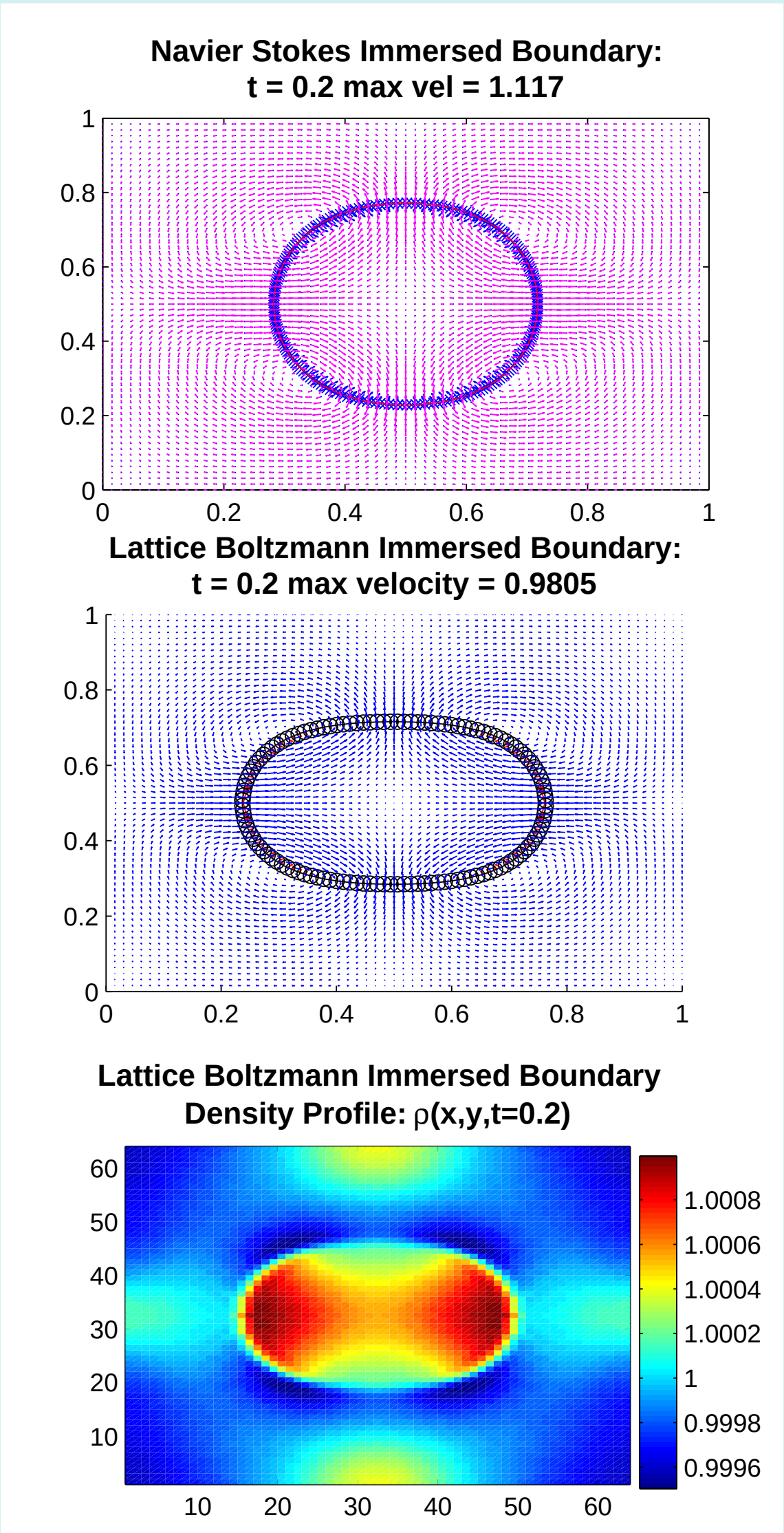


- For our lattice Boltzmann-Immersed Boundary simulations, convergence in space and time is first order, most likely due to the thinness of the elastic membrane.

Comparison to Navier Stokes



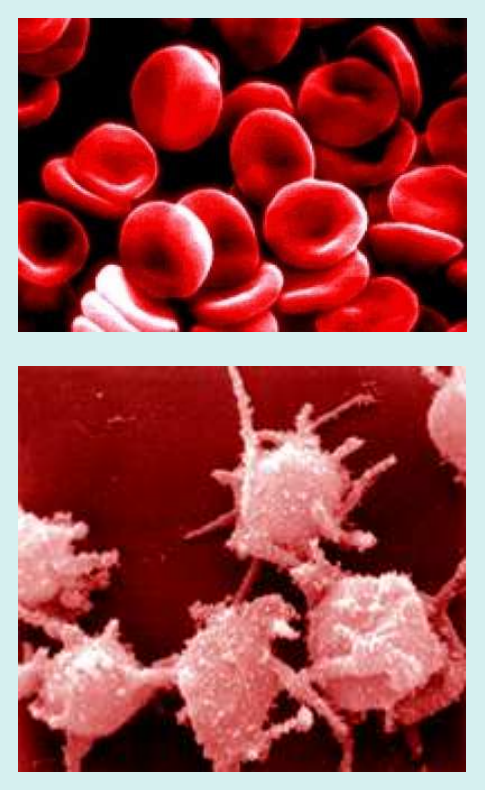
Space discretization: $dx = \frac{1}{64}$,
Time discretization$dt = \frac{1}{640}$.



Space discretization: $dx = \frac{1}{64}$,
Time discretization$dt = \frac{1}{5120}$.

- LBM is compressible, but as the Mach number goes to zero ($c = \frac{dx}{dt} >> \|\mathbf{u}\|$), the fluid approaches an incompressible limit.
- LBM fluid velocity appears slightly slower than Navier Stokes fluid velocity.

Future Work



Courtesy of Dr. Elisabeth Maurer-Spurej.

- Compare LBM and Navier Stokes for IB problem in more detail.
- Test speed of current LBM.
- Parallelize LBM for simulating particles in flow.
- Simulate flow of platelets in blood vessels.

References

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