1. (4 points)

(a) Find the characteristic polynomial of the matrix

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}.$$ 

$$f_{A}(\lambda) = \det (A - \lambda I_{2}) = \det \left( \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = \det \begin{pmatrix} 1 - \lambda & -1 \\ 2 & 4 - \lambda \end{pmatrix}$$

$$= (1-\lambda)(4-\lambda) + 2 = 4 - 5\lambda + \lambda^2 + 2$$

$$= \lambda^2 - 5\lambda + 6$$

Factor

$$f_{A}(\lambda) = (\lambda - 2)(\lambda - 3)$$

(b) Determine the eigenvalues of $A$.

$$f_{A}(\lambda) = 0 \implies \lambda = 2 \text{ or } \lambda = 3.$$ Therefore, 2 and 3 are the eigenvalues of $A$. 

2. (3 points) Let

\[ A = \begin{pmatrix} a & k \\ -1 & a \end{pmatrix}. \]

For which values of \( k \) does the matrix \( A \) have no (real) eigenvalues?

\[ \det \begin{pmatrix} a - \lambda & k \\ -1 & a - \lambda \end{pmatrix} = (a - \lambda)^2 + k = 0 \implies \]

\[ (a - \lambda)^2 = -k \]

always positive

\[ \implies \text{ we need } -k > 0 \text{ or } k < 0 \text{ to have real eigenvalues} \]

So, for \( k > 0 \) there are no real eigenvalues.

3. (4 points) True or false. Indicate whether the following statements are true or false.

(a) There exists a \( 3 \times 3 \) matrix \( A \) without any real eigenvalues.

\[ \text{False: The char. poly. will always have at least one real root} \]

(b) A square matrix \( A \) is invertible if and only if 0 is not an eigenvalue of \( A \).

\[ \text{True: } 0 \text{ is not an eigenvalue if and only if } \ker(A) = \{0\} \text{ if and only if } A \text{ is invertible.} \]