1. (4 points) Find the determinant of the following matrix. Show your work.

\[
\begin{pmatrix}
0 & 3 & 1 & 0 \\
2 & 0 & 0 & 0 \\
0 & 1 & 4 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

Expand across the bottom row:

\[
1 \cdot \det \begin{pmatrix}
0 & 3 & 1 \\
2 & 0 & 0 \\
0 & 1 & 4 \\
\end{pmatrix} = \text{(expand down first column)}
\]

\[
= 1 \cdot (-2) \cdot \det \begin{pmatrix}
3 & 1 \\
1 & 4 \\
\end{pmatrix} = 1 \cdot (-2) \cdot (12 - 1) = 1 \cdot (-2) \cdot 11 = -22
\]
2. (6 points) Consider a $3 \times 3$ matrix $A$ with rows $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$

$$A = \begin{pmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \end{pmatrix}$$

and suppose $\det(A) = 8$.

(a) Find $\det \begin{pmatrix} 2\vec{v}_1 \\ \vec{v}_3 \\ \vec{v}_2 \end{pmatrix}$.

- Multiplying the 1st row by 2 scales $\det(A)$ by 2.
- Swapping rows 2 and 3 scales $\det(A)$ by -1.

\[ \therefore \quad \text{we get } (-1) \cdot 2 \cdot \det(A) = [-16] \]

(b) Find $\det(A^{-1}A^T A)$.

\[
\det(A^{-1}A^T A) = \det(A^{-1}) \cdot \det(A^T) \cdot \det(A) \\
= \det(A^{-1}) \cdot \det(A^T) \cdot \det(A) \\
= \det(A^T) = \det(A) = [8] 
\]

(c) Suppose the vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ are orthogonal and have norm $x$, i.e.

$$\|\vec{v}_1\| = \|\vec{v}_2\| = \|\vec{v}_3\| = x.$$ 

Find $x$.

Since $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are orthogonal, they span a cube in $\mathbb{R}^3$.

We know this cube has volume $\det(A) = 8$, so each side has length $[2]$. 

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3. (8 points) For $\alpha \in \mathbb{R}$, let

$$
A = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 2 & \alpha \\
0 & 0 & 0 & 0 & 2
\end{pmatrix}
$$

and observe that the only eigenvalues of $A$ are $\lambda = 1$ and $\lambda = 2$.

(a) What is the algebraic multiplicity of the eigenvalue $\lambda = 1$?

$$
\text{geom. mult} = \dim \ker (A - I) = \dim \ker \begin{pmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 2 & \alpha \\
0 & 0 & 0 & 0 & 2
\end{pmatrix}
= 1
$$

(b) What is the geometric multiplicity of the eigenvalue $\lambda = 1$?

(c) For which values of $\alpha$ will the geometric multiplicity of $\lambda = 2$ and the algebraic multiplicity of $\lambda = 2$ be equal?

Need $\dim \ker \begin{pmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 2 & \alpha \\
0 & 0 & 0 & 0 & 2
\end{pmatrix} = 2$. This is possible if and only if there are two rows of zeros, or $\alpha = 0$.

(d) If $\alpha = 0$, is $A$ diagonalizable? Explain your answer.

No, since by (a) and (b) there is an eigenvalue whose alg. mult. is strictly less than its alg. mult.
4. (8 points) Suppose \[ A = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}. \]

Find a diagonal matrix \( D \) and an orthogonal matrix \( S \) such that \( D = S^T A S \). Show your work.

\[ \rho_A(x) = x^2 - 2x = (x-5)(x+5) \]

\[ E_5 = \text{ker} \begin{pmatrix} -2 & 4 \\ 4 & -8 \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \]

\[ E_{-5} = \text{ker} \begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\} \]

\[ D = \begin{pmatrix} 5 & 0 \\ 0 & -5 \end{pmatrix} \quad \text{and} \quad S = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \]
5. (10 points)

(a) Is the matrix \( \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \) similar to the matrix \( \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} \)?

\[ \text{No} \] they don't have the same trace.

(b) Is the matrix \( \begin{pmatrix} -1 & 6 \\ -2 & 6 \end{pmatrix} \) similar to the matrix \( \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \)?

\[ f_\lambda(x) = x^2 - 5x + 6 = (x-2)(x-3) \]

\[ \text{Yes} \] because they have the same eigenvalues and they are all distinct.

(c) Suppose \( \vec{v} \) is an eigenvector of an invertible \( n \times n \) matrix \( A \) and let \( \lambda \) be the associated eigenvalue. Is \( \vec{v} \) an eigenvector of \( A^{-1} + 2I_n \)? If so, what is the associated eigenvalue? If not, explain why.

\[ \begin{pmatrix} A^{-1} + 2I_n \end{pmatrix} \vec{v} = A^{-1} \vec{v} + 2I_n \vec{v} = \frac{1}{\lambda} \vec{v} + 2 \vec{v} = \frac{1}{\lambda + 2} \vec{v} \]

Therefore, \( \vec{v} \) is an eigenvector \( w \) eigenvalue \( \frac{1}{\lambda + 2} \).

(d) Let \( q(x_1, x_2, x_3) = 3x_1^2 + 4x_2^2 + 5x_3^2 - 6x_1x_2 + 7x_2x_3 \) be a quadratic form. Find a symmetric \( 3 \times 3 \) matrix \( A \) such that \( q(\vec{x}) = \vec{x}^T A \vec{x} \).

\[ A = \begin{pmatrix} 3 & -3 & 0 \\ -3 & 4 & \frac{7}{2} \\ 0 & \frac{7}{2} & 5 \end{pmatrix} \]
(e) Suppose $A$ is an orthogonal $n \times n$ matrix. Find $\text{trace}(A^2)$.

The eigenvalues of $A$ are $\pm 1$, so the eigenvalues of $A^2$ are all $(\pm 1)^2 = 1$. Therefore, the trace of $A^2$ is the sum of the eigenvalues $= \boxed{n}$.

6. (4 points) Multiple choice. Choose the best answer to each of the following questions.

(a) If a $2 \times 2$ matrix $A$ has eigenvalues $\lambda = 1$ and $\lambda = 2$, then $\text{trace}(A^{100})$ is

i. $2^{100}$

ii. $2^{101}$

iii. $2^{100} + 1$

iv. $3^{100}$

v. Cannot be determined from the given information.

(b) If $A$ is a symmetric positive definite $2 \times 2$ matrix, which of the following must be true?

i. The algebraic multiplicity of every eigenvalue is 1.

ii. $A^{-1}$ is negative definite.

iii. $\det(-A) < 0$

iv. The curve defined by the equation $\vec{x}^T A \vec{x} = 1$ is an ellipse.

v. none of the above.