1. (4 points) Find the determinant of the following matrix. Show your work.

\[
\begin{pmatrix}
0 & 3 & 1 & 0 \\
2 & 0 & 0 & 0 \\
0 & 1 & 4 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}.
\]
2. (6 points) Consider a $3 \times 3$ matrix $A$ with rows $\vec{v}_1, \vec{v}_2, \vec{v}_3$

$$A = \begin{pmatrix} - \vec{v}_1 & - \\ - \vec{v}_2 & - \\ - \vec{v}_3 & - \end{pmatrix}$$

and suppose $\det(A) = 8$.

(a) Find $\det \begin{pmatrix} 2\vec{v}_1 \\ \vec{v}_3 \\ \vec{v}_2 \end{pmatrix}$.

(b) Find $\det(A^{-1}A^T)$.

(c) Suppose the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are orthogonal and have norm $x$, i.e.

$$\|\vec{v}_1\| = \|\vec{v}_2\| = \|\vec{v}_3\| = x.$$ 

Find $x$.  

2
3. (8 points) For $\alpha \in \mathbb{R}$, let

$$A = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 2 & \alpha \\
0 & 0 & 0 & 0 & 2
\end{pmatrix}$$

and observe that the only eigenvalues of $A$ are $\lambda = 1$ and $\lambda = 2$.

(a) What is the algebraic multiplicity of the eigenvalue $\lambda = 1$?

(b) What is the geometric multiplicity of the eigenvalue $\lambda = 1$?

(c) For which values of $\alpha$ will the geometric multiplicity of $\lambda = 2$ and the algebraic multiplicity of $\lambda = 2$ be equal?

(d) If $\alpha = 0$, is $A$ diagonalizable? Explain your answer.
4. (8 points) Suppose

\[ A = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}. \]

Find a diagonal matrix \( D \) and an orthogonal matrix \( S \) such that \( D = S^T A S \). Show your work.
5. (10 points)

(a) Is the matrix \[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{pmatrix}
\]
similar to the matrix \[
\begin{pmatrix}
2 & 2 & 2 \\
2 & 2 & 2 \\
2 & 2 & 2 \\
\end{pmatrix}
\]?

(b) Is the matrix \[
\begin{pmatrix}
-1 & 6 \\
-2 & 6 \\
\end{pmatrix}
\]
similar to the matrix \[
\begin{pmatrix}
3 & 0 \\
0 & 2 \\
\end{pmatrix}
\]?

(c) Suppose \( \vec{v} \) is an eigenvector of an invertible \( n \times n \) matrix \( A \) and let \( \lambda \) be the associated eigenvalue. Is \( \vec{v} \) an eigenvector of \( A^{-1} + 2I_n \)? If so, what is the associated eigenvalue? If not, explain why.

(d) Let \( q(x_1, x_2, x_3) = 3x_1^2 + 4x_2^2 + 5x_3^2 - 6x_1x_2 + 7x_2x_3 \) be a quadratic form. Find a symmetric \( 3 \times 3 \) matrix \( A \) such that \( q(\vec{x}) = \vec{x}^T A \vec{x} \).
(e) Suppose $A$ is a symmetric, orthogonal $n \times n$ matrix. Find the eigenvalues (with multiplicities) of $A^2$.

6. (4 points) Multiple choice. Choose the best answer to each of the following questions.

(a) If a $2 \times 2$ matrix $A$ has eigenvalues $\lambda = 1$ and $\lambda = 2$, then trace($A^{100}$) is

i. $2^{100}$

ii. $2^{101}$

iii. $2^{100} + 1$

iv. $3^{100}$

v. Cannot be determined from the given information.

(b) If $A$ is a symmetric positive definite $2 \times 2$ matrix, which of the following must be true?

i. The algebraic multiplicity of every eigenvalue is 1.

ii. $A^{-1}$ is negative definite.

iii. det($-A$) < 0

iv. The curve defined by the equation $\vec{x}^T A \vec{x} = 1$ is an ellipse.

v. none of the above.