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Math 5760 - Intro Math Finance
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Solutions to homework #9 Dec. 8, 2009

Problem 1.

(a) The density of a normal random variable X $\mathcal{N}(\mu, \beta^2)$ is given by

$$\phi(x) = \frac{1}{\sqrt{2\pi}\beta} \exp\left(-\frac{(x-\mu)^2}{2\beta^2}\right).$$

Hence, using the change of variables $y = \frac{x-\mu}{\sigma}$,

$$\begin{aligned}
\mathbb{E}[e^X] &= \frac{1}{\sqrt{2\pi}\beta} \int_{-\infty}^{\infty} e^x e^{-\frac{(x-\mu)^2}{2\beta^2}} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\mu+\beta y} e^{-\frac{y^2}{2}} dy \\
&= e^{\mu} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(y^2-2\beta y+\beta^2)} e^{\frac{\beta^2}{2}} dy \\
&= e^{\mu+\frac{\beta^2}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(y-\beta)^2}{2}} dy \\
&= e^{\mu+\frac{\beta^2}{2}}.
\end{aligned}$$

(b) By the formula seen in class, under $\tilde{\mathbb{P}}$,

$$\lim_{N\to\infty} S(T) = \exp\left(\ln(S_0) + \left(R - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}Z\right),$$

where Z is a standard normal random variable. Hence, $\lim_{N\to\infty} S(T) = \exp(X)$, where X is a normal random variable with parameters $\mu = \ln(S_0) + \left(R - \frac{\sigma^2}{2}\right)T$ and $\beta = \sigma\sqrt{T}$. Then, by (a),

$$\tilde{\mathbb{E}}[\lim_{N\to\infty} S(T)] = \tilde{\mathbb{E}}[e^X] = \exp\left(\ln(S_0) + \left(R - \frac{\sigma^2}{2}\right)T + \frac{\sigma^2 T}{2}\right) = S_0 e^{RT}.$$

(c) After N periods, an amount S_0 invested on the money market becomes

$$S_0(1+r)^N = S_0\left(1 + \frac{RT}{N}\right)^N \longrightarrow S_0 e^{RT},$$

as N goes to ∞. Hence, the money market and the stock market share the same average behavior under the risk-neutral probability measure, as was the case in the binomial model.

Problem 2.

Throughout this problem, we will write ϕ the derivative of Φ (i.e. the standard normal density). Let's check what happens with the derivatives of d_1 and d_2. We have

$$\frac{\partial d_2}{\partial T} = \frac{\partial}{\partial T}(d_1 - \sigma\sqrt{T}) = \frac{\partial d_1}{\partial T} - \frac{\sigma}{2\sqrt{T}}.$$

and

$$\frac{\partial d_2}{\partial s} = \frac{\partial d_1}{\partial s}.$$

Moreover,

$$\phi(d_2) = \phi(d_1 - \sigma\sqrt{T}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(d_1-\sigma\sqrt{T})^2}{2}} = \phi(d_1) e^{\frac{\sigma^2 T}{2}} e^{\sigma\sqrt{T} d_1} = \frac{S_0}{K} e^{RT}\phi(d_1).$$

We now would like to consider the function $(s,T) \mapsto C(s,T,K,\sigma)$. We have :

$$\begin{aligned}
\frac{\partial C}{\partial T}(s,T,K,\sigma) &= s\phi(d_1)\frac{\partial d_1}{\partial T} + KRe^{-RT}\Phi(d_2) - Ke^{-RT}\phi(d_2)\frac{\partial d_2}{\partial T} \\
&= KRe^{-RT}\Phi(d_2) + s\phi(d_1)\left(\frac{\partial d_1}{\partial T} - \frac{\partial d_2}{\partial T}\right) \\
&= KRe^{-RT}\Phi(d_2) + \frac{s\sigma}{2\sqrt{T}}\phi(d_1),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial C}{\partial s}(s,T,K,\sigma) &= \Phi(d_1) + s\phi(d_1)\frac{\partial d_1}{\partial s} - Ke^{-RT}\phi(d_2)\frac{\partial d_2}{\partial s} \\
&= \Phi(d_1) + s\phi(d_1)\left(\frac{\partial d_1}{\partial s} - \frac{\partial d_2}{\partial s}\right) \\
&= \Phi(d_1)
\end{aligned}$$

and

$$\frac{\partial^2 C}{\partial s^2}(s,T,K,\sigma) = \phi(d_1)\frac{\partial d_1}{\partial s}.$$

If we now consider the function u, we have $u(s,t) = C(s, T-t, K, \sigma)$. Hence,

$$\frac{\partial u}{\partial t}(s,t) = -\frac{\partial C}{\partial T}(s, T-t, K, \sigma) = -KRe^{-R(T-t)}\Phi(d_2) - \frac{s\sigma}{2\sqrt{T-t}}\phi(d_1),$$

$$\frac{\partial u}{\partial s}(s,t) = \frac{\partial C}{\partial s}(s, T-t, K, \sigma) = \Phi(d_1)$$

and

$$\frac{\partial^2 u}{\partial s^2}(s,t) = \frac{\partial^2 C}{\partial s^2}(s, T-t, K, \sigma) = \phi(d_1)\frac{\partial d_1}{\partial s},$$

where d_1 and d_2 are now functions of $T-t$.

We also have

$$\frac{\partial d_1}{\partial s} = \frac{1}{\sigma\sqrt{T}}\frac{K}{s}\frac{1}{K} = \frac{1}{s\sigma\sqrt{T}}.$$

Hence,

$$\begin{aligned}
&\frac{\partial u}{\partial t} + Rs\frac{\partial u}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 u}{\partial s^2} \\
&= -KRe^{-R(T-t)}\Phi(d_2) - \frac{s\sigma}{2\sqrt{T-t}}\phi(d_1) + Rs\Phi(d_1) + \frac{1}{2}\sigma^2 s^2 \phi(d_1)\frac{\partial d_1}{\partial s} \\
&= -KRe^{-R(T-t)}\Phi(d_2) - \frac{s\sigma}{2\sqrt{T-t}}\phi(d_1) + Rs\Phi(d_1) + \frac{s\sigma}{2\sqrt{T-t}}\phi(d_1) \\
&= Rs\Phi(d_1) - KRe^{-R(T-t)}\Phi(d_2) \\
&= RC(s, T-t, K, \sigma) = Ru.
\end{aligned}$$

Hence, the PDE is satisfied. It remains to check the terminal condition. We have

$$\begin{aligned}
\lim_{T\to 0} d_2 = \lim_{T\to 0} d_1 &= \lim_{T\to 0} \frac{1}{\sigma}\ln\left(\frac{S_0}{K}\right)\frac{1}{\sqrt{T}} + \frac{1}{\sigma}(R + \frac{\sigma^2}{2})\sqrt{T} \\
&= \lim_{T\to 0} \frac{1}{\sigma}\ln\left(\frac{S_0}{K}\right)\frac{1}{\sqrt{T}} = \begin{cases} +\infty & \text{if } S_0 > K \\ 0 & \text{if } S_0 = K \\ -\infty & \text{if } S_0 < K \end{cases}
\end{aligned}$$

Then,

$$\begin{aligned}
u(s,T) &= \lim_{T\to 0} C(s, T, K, \sigma) = \lim_{T\to 0} S_0\Phi(d_1) - K\Phi(d_2) \\
&= \begin{cases} (S_0 - K)\Phi(+\infty) & \text{if } S_0 > K \\ 0 & \text{if } S_0 = K \\ (S_0 - K)\Phi(-\infty) & \text{if } S_0 < K \end{cases} \\
&= \begin{cases} S_0 - K & \text{if } S_0 > K \\ 0 & \text{if } S_0 \leqslant K \end{cases} = (S_0 - K)^+
\end{aligned}$$

Problem 3.

(a) We want to use Black-Scholes formula with the following parameters :

$$S_0 = 100;\ K = 120;\ R = 0.02;\ \sigma = 0.2;\ T = 2$$

This gives us (with the notation of Problem 2):

$$V_0 = C(100, 2, 120, 0.2) = 5.94694.$$

(b) We will use the following formulas :

$$r = \frac{RT}{N};\ u = \exp\left((R - \frac{\sigma^2}{2})\frac{T}{N} + \sigma\sqrt{\frac{T}{N}}\right);\ d = \exp\left((R - \frac{\sigma^2}{2})\frac{T}{N} - \sigma\sqrt{\frac{T}{N}}\right),$$

so that $\tilde{p} = \tilde{q} = \frac{1}{2}$. Hence, we obtain

$\mathbf{N = 50}$	$\mathbf{N = 100}$	$\mathbf{N = 200}$
$r = 0.0008$	$r = 0.0004$	$r = 0.0002$
$u = 1.040811$	$u = 1.028688$	$u = 1.020201$
$d = 0.9607894$	$d = 0.972112$	$d = 0.9801987$

In any case, $\tilde{p} = \tilde{q} = \frac{1}{2}$.

(c) Now, running the algorithm from Homework #3 with the parameters in (b), we find the following prices.

N = 50	**N = 100**	**N = 200**
$V_0 = 5.965244$	$V_0 = 5.950366$	$V_0 = 5.956822$

We notice that the approximation is pretty good for all cases. We could expect to see the price get closer to the Black-Scholes formula as N increases. Although we know that this converges as N goes to ∞, the sequence is not decreasing, hence the fact that $N = 200$ is actually not as good as $N = 100$.

References

[1] Shreve S.E. *Stochastic Calculus for Finance I : the Binomial Asset Pricing Model.* Springer, 2005.