Additional Chapter : Insight on Continous-Time Models and Statistical Aspect.

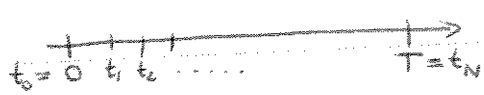
In this chapter, we would like to study what happens if the number of periods in a binomial asset pricing model gets very large. Moreover, we will give a slight introduction on how to fit a binomial model to reality.

Suppose we study a European-type derivative security on a stock for which we know the past prices up to the current date $t=0$. If this stock follows a binomial model, we know how to price this derivative security.

We are going to consider an European call on some stock S with strike price K and maturity T. Suppose that the current stock price is S_0. We want to use an N-period binomial model. So we will consider that the time-period $[0,T]$ is divided in N smaller periods by setting

$$\Delta t = \frac{T}{N}$$

$$t_n := n\,\Delta t = n\frac{T}{N} \qquad \text{for } n = 0, \dots, N.$$



Now, we will consider the discrete-time stock price $S_n := S(t_n)$ to evolve according to a binomial model. We frist need to identify the parameters r, u and d. (We may also want to find p and q, but they won't be needed for pricing purposes.)

First study what is going on with r.

Money market : (parameter r)

Suppose that the interest rate on the market we are studying is given by R, in the sense that \$1 invested on the money market becomes $(1+R)$ after one unit of time. Then, on the time period $[0,T]$, \$1 becomes $(1+R)^T$. Now, if we take the interst rate over one period to be $r = \frac{RT}{N}$, we get

$$(1+r)^N = \left(1+\frac{RT}{N}\right)^N \xrightarrow[N\to\infty]{} e^{RT} \simeq (1+R)^T$$

if R is small, which is a reasonable assumption.

$\longrightarrow$ This would give us the same amount at time T approximatively.

Stock market.

We suppose that we know the price history of our stock. Hence, using these data, we can estimate empirical parameters of our stock in order to catch properties of its behavior. We usually consider the average growth rate μ and the volatility σ of the stock, defined respectively by

$$E\left[\frac{S_{n+1}-S_n}{S_n}\right] = \mu\,\Delta t \quad \text{and} \quad E\left[\left(\frac{S_{n+1}-S_n}{S_n}-\mu\,\Delta t\right)\right] = \sigma^2\,\Delta t$$

Now, using our history of the price, we can take prices at times separated by an interval $\frac{1}{N}$ and build estimates for these parameters using

$$\frac{1}{M}\sum_{n=0}^{M-1}\frac{S_{n+1}-S_n}{S_n} = \hat{\mu}\,\Delta t$$

$$\frac{1}{M-1}\sum_{n=0}^{M-1}\left(\frac{S_{n+1}-S_n}{S_n}-\hat{\mu}\,\Delta t\right) = \hat{\sigma}^2\,\Delta t$$

if our data consists of a size M sample of stock prices over size Δt intervals.

Now we would like to find u and d such that our model fits with these estimates.

Actual model: We can consider without loss of generality that we would like $p = q = 1/2$. Let u and d be of the form

$$u = e^{a\frac{T}{N} + b\sqrt{\frac{T}{N}}} \quad ; \quad d = e^{a\frac{T}{N} - b\sqrt{\frac{T}{N}}}$$

We recall that $r = \frac{RT}{N}$ over one period. We can assume that N is large, hence we can obtain approximations for such u and d by Taylor's formula.

- $$u = e^{a\frac{T}{N} + b\sqrt{\frac{T}{N}}} \simeq 1 + \left(a\frac{T}{N} + b\sqrt{\frac{T}{N}}\right) + \frac{1}{2}\left(a\frac{T}{N} + b\sqrt{\frac{T}{N}}\right)^2 + \dots$$

$$\simeq 1 + a\frac{T}{N} + b\sqrt{\frac{T}{N}} + \frac{b^2}{2}\frac{T}{N} + O\left(\frac{1}{N^{3/2}}\right)$$

$$= 1 + \left(a + \frac{b^2}{2}\right)\frac{T}{N} + b\sqrt{\frac{T}{N}} + O\left(\frac{1}{N^{3/2}}\right)$$

This means "terms of order larger than 3/2 that we neglect".

- (similarly) $$d \cong 1 + \left(a + \frac{b^2}{2}\right)\frac{T}{N} - b\sqrt{\frac{T}{N}} + O\left(\frac{1}{N^{3/2}}\right)$$

These are order 1 approximations of u and d in terms of $1/N$.

Now, for a binomial model,

$$E\left[\frac{S_{n+1} - S_n}{S_n}\right] = p(u-1) + q(d-1) = pu + qd - 1.$$

Hence, we need (here, $p = q = 1/2$) : $$\frac{u+d}{2} - 1 = \hat{\mu}\Delta t = \hat{\mu}\frac{T}{N}$$

The approximations above lead to $\left(a+\frac{b^2}{2}\right)\frac{T}{N} = \hat{\mu}\frac{T}{N}$ or $\hat{\mu} = a + \frac{b^2}{2}$

Further, $E\left[\left(\frac{S_{n+1}-S_n}{S_n} - \hat{\mu}\Delta t\right)^2\right] = p(u-1-\hat{\mu}\Delta t)^2 + q(d-1-\hat{\mu}\Delta t)^2$

But, $u-1-\hat{\mu}\Delta t = u-1-\left(a+\frac{b^2}{2}\right)\frac{T}{N} \cong b\sqrt{\frac{T}{N}} + O\left(\frac{1}{N^{3/2}}\right)$

$d-1-\hat{\mu}\Delta t = -b\sqrt{\frac{T}{N}} + O\left(\frac{1}{N^{3/2}}\right)$

$$\Rightarrow E\left[\left(\frac{S_{n+1}-S_n}{S_n} - \hat{\mu}\Delta t\right)^2\right] \simeq b^2\frac{T}{N} + O\left(\frac{1}{N^2}\right) = \hat{\sigma}^2\frac{T}{N}$$

$$\Rightarrow b = \hat{\sigma} \quad \Rightarrow \quad a = \hat{\mu} - \frac{\hat{\sigma}^2}{2}$$

Hence, taking $u = e^{\left(\hat{\mu}-\frac{\hat{\sigma}^2}{2}\right)\frac{T}{N} + \hat{\sigma}\sqrt{\frac{T}{N}}}$, $d = e^{\left(\hat{\mu}-\frac{\hat{\sigma}^2}{2}\right)\frac{T}{N} - \hat{\sigma}\sqrt{\frac{T}{N}}}$, $r = \frac{RT}{N}$ gives a model that fits with the history of the stock. In that case, $p=q=1/2$ and

$$\tilde{p} = \frac{1+\frac{RT}{N}-d}{u-d} \simeq \frac{\cancel{1}+\frac{RT}{N} - \left(\cancel{1}+\hat{\mu}\frac{T}{N} - \hat{\sigma}\sqrt{\frac{T}{N}}\right)}{2\hat{\sigma}\sqrt{\frac{T}{N}}}$$

$$= \frac{1}{2} + \left(\frac{R}{2\hat{\sigma}} - \frac{\hat{\mu}}{2\hat{\sigma}}\right)\sqrt{\frac{T}{N}}$$

$$\tilde{q} \simeq \frac{1}{2} - \left(\frac{R}{2\hat{\sigma}} - \frac{\hat{\mu}}{2\hat{\sigma}}\right)\sqrt{\frac{T}{N}}$$

Risk-Neutral model.

We could also want to fit a model with respect to risk-neutral measure. In that case, we want the average growth rate to match the one of the money market, namely r. So, we want to fit

$$\tilde{E}\left[\frac{S_{n+1}-S_n}{S_n}\right] = R\frac{T}{N} = R\Delta t \qquad E\left[\left(\frac{S_{n+1}-S_n}{S_n} - r\Delta t\right)^2\right] = \sigma^2\Delta t = \sigma^2\frac{T}{N}.$$

If we want those two equations to be satisfied, it comes back to taking $\hat{\mu} = r$ above and we get

$$u = e^{(R-\frac{\sigma^2}{2})\frac{T}{N} + \sigma\sqrt{\frac{T}{N}}} \qquad d = e^{(R-\frac{\sigma^2}{2})\frac{T}{N} - \sigma\sqrt{\frac{T}{N}}}$$

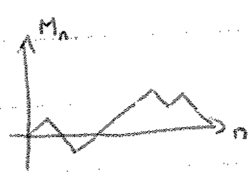
$$\tilde{p} = \tilde{q} = 1/2.$$

That's the model that we will consider in the sequel.

<u>Random Walk</u>: we now want to define a <u>random walk</u> stochastic process.

For $i = 1, \dots, N$, let
$$X_i(\omega_i) = \begin{cases} 1 & \text{if } \omega_i = H \\ -1 & \text{if } \omega_i = T \end{cases}$$

Let $M_n = \sum_{i=1}^{n} X_i \quad (= \#H - \#T \text{ in } \omega_1 \cdots \omega_n)$



Now, by definition of the stock price in the binomial model, we have

$$S(t_n) = S_n = S_0 \cdot u^{\#H \text{ in } \omega_1 \cdots \omega_n} d^{\#T \text{ in } \omega_1 \cdots \omega_n}$$

$$= S_0 \, e^{(R-\frac{\sigma^2}{2})\frac{T}{N} n} \, e^{\sigma\sqrt{\frac{T}{N}}(\#H - \#T)}$$

$$= S_0 \, e^{(R-\frac{\sigma^2}{2})t_n} \, e^{\sigma\sqrt{\frac{T}{N}} M_n} = S_0 \, e^{(R-\frac{\sigma^2}{2})t_n} \, e^{\sigma\sqrt{t_n}\frac{M_n}{\sqrt{n}}}$$

where M_n is defined above.

In order to price a European derivative security, only $S(T) = S(t_N) = S_N$ is relevant. We have:

$$S(T) = S(t_N) = S_N = S_0\, e^{(R-\frac{\sigma^2}{2})t_N}\, e^{\sigma\sqrt{t_N}\frac{M_N}{\sqrt{N}}}$$
$$= S_0\, e^{(R-\frac{\sigma^2}{2})T}\, e^{\sigma\sqrt{T}\frac{M_N}{\sqrt{N}}}$$

We now would like to know what happens when $N \to \infty$

M_N is the sum of N iid random variables $(X_1 \cdots X_N)$, each with

$$E[X_i] = E[X_1] = \tilde{p}\cdot 1 + \tilde{q}(-1) = \tilde{p} - \tilde{q} = 1/2 - 1/2 = 0$$

and $E[X_i^2] = E[X_1^2] = \tilde{p}\cdot 1^2 + \tilde{q}(-1)^2 = \tilde{p} + \tilde{q} = 1$

Hence, by the Central Limit Theorem, $\frac{M_N}{\sqrt{N}} \xrightarrow[N\to\infty]{} Z$, where Z is a $\mathcal{N}(0,1)$ random variable.

$$\Rightarrow \boxed{S(T) = S_0\, e^{(R-\frac{\sigma^2}{2})T + \sigma\sqrt{T}Z}}$$

We say that $S(T)$ has a log-normal distribution, because
$\ln(S(T)) = \ln(S_0) + (R - \frac{\sigma^2}{2})T + \sigma\sqrt{T}Z$ has a normal distribution. (*)

We will see next term that this what we obtain using continuous-time models.

(*) To be precise, $\ln(S(T)) \sim \mathcal{N}\left(\ln(S_0) + (R-\frac{\sigma^2}{2})T \,;\, \sigma^2 T\right)$

Pricing formula: By the risk-neutral pricing formula from Chapter 2, we know that

$$V_0 = \tilde{E}\left[\frac{V_N}{(1+r)^N}\right] = \tilde{E}\left[\frac{\max(S_N - K, 0)}{(1+\frac{RT}{N})^N}\right]$$
$$\xrightarrow[N\to\infty]{} \tilde{E}\left[e^{-RT}\max\left(S_0 e^{(R-\frac{\sigma^2}{2})T + \sigma\sqrt{T}Z} - K, 0\right)\right]$$

$$\Rightarrow \boxed{V_0 = e^{-RT}\,\tilde{E}\left[\max\left(S_0 e^{(R-\frac{\sigma^2}{2})T + \sigma\sqrt{T}Z} - K, 0\right)\right]}$$

Under $\tilde{\mathbb{P}}$, we know that $Z \sim \mathcal{N}(0,1)$. That's why we have:

$$V_0 = e^{-RT}\int_{-\infty}^{\infty} \max\left(S_0 e^{(R-\frac{\sigma^2}{2})T+\sigma\sqrt{T}Z} - K,\ 0\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{Z^2}{2}}\,dZ$$

But $S_0 e^{(R-\frac{\sigma^2}{2})T+\sigma\sqrt{T}z} \geq K \iff Z \geq \frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{K}{S_0}\right) - \left(R-\frac{\sigma^2}{2}\right)T\right)$

$$\Longrightarrow V_0 = e^{-RT}\int_{\frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{K}{S_0}\right)-\left(R-\frac{\sigma^2}{2}\right)T\right)}^{+\infty}\left(S_0 e^{(R-\frac{\sigma^2}{2})T+\sigma\sqrt{T}z} - K\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}\,dz$$

$$= e^{-RT}\int_{\frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{K}{S_0}\right)-\left(R-\frac{\sigma^2}{2}\right)T\right) = \alpha}^{+\infty} S_0 e^{(R-\frac{\sigma^2}{2})T} e^{\sigma\sqrt{T}z}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}\,dz - Ke^{-RT}\int_{\frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{K}{S_0}\right)-\left(R-\frac{\sigma^2}{2}\right)T\right)}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}\,dz$$

$$= S_0 \cancel{e^{-\frac{\sigma^2}{2}T}}\frac{1}{\sqrt{2\pi}}\int_{\alpha}^{+\infty} e^{\underbrace{-\frac{1}{2}(z^2-2\sigma\sqrt{T}z+\sigma^2T)}_{\frac{y^2}{2}\ \iff\ y=(z-\sigma\sqrt{T})}}\cancel{e^{\frac{\sigma^2}{2}T}}\,dz - Ke^{-RT}\int_{\alpha}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}\,dz$$

$$= S_0\frac{1}{\sqrt{2\pi}}\int_{\alpha-\sigma\sqrt{T}}^{+\infty} e^{-\frac{y^2}{2}}\,dy - Ke^{-RT}\int_{\alpha}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}$$

$$= S_0\left(1-\Phi(\alpha-\sigma\sqrt{T})\right) - Ke^{-RT}\left(1-\Phi(\alpha)\right)$$

$$= S_0\,\Phi(-\alpha+\sigma\sqrt{T}) - Ke^{-RT}\,\Phi(-\alpha)$$

$$\boxed{V_0 = S_0\,\Phi(d_1) - Ke^{-RT}\,\Phi(d_2)}$$

with $d_1 = \sigma\sqrt{T} - \frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{K}{S_0}\right) - \left(R-\frac{\sigma^2}{2}\right)T\right) = \frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{S_0}{K}\right) + \left(R+\frac{\sigma^2}{2}\right)T\right)$

$d_2 = d_1 - \sigma\sqrt{T}$.

This formula is the well-known <u>Black-Scholes formula</u>.

<u>Example.</u> In HW \#3 we computed the price of a European call with strike $K = \$110$ and maturity $N = 100$ with $S_0 = \$100$, $r = 0.001$. If we suppose that $T = 1$, then $R = \frac{Nr}{T} = 0{,}1 = 10\%$ over the time period.

Then, B-S formula needs σ. We can compute it as

$$\sigma^2 = \tilde{E}\left[\left(\frac{S_1}{S_0} - E\left[\frac{S_1}{S_0}\right]\right)^2\right] = \tilde{p}\,(u-(1+r))^2 + \tilde{q}\,(d-(1+r))^2$$
$$\cong 4 \cdot 10^{-4}$$

Then, replacing in B-S formula, we obtain

$V_0 = 8.1731$

The algorithm gave 8.1644, hence the approximation is very good.

Example: behavior of $V_0(K)$. We want to compute $\frac{\partial V_0(K)}{\partial K}$. As intuition indicates, V_0 should be decreasing with respect to K.

- $\frac{\partial d_1}{\partial K} = \frac{1}{\sigma\sqrt{T}} \frac{K}{S_0} \left(-\frac{S_0}{K^2}\right) = \frac{-1}{\sigma\sqrt{T}\,K}$ $\qquad \frac{\partial d_2}{\partial K} = \frac{\partial d_1}{\partial K}$.

- $\frac{dV_0}{dK} = S_0\,\varphi(d_1)\frac{\partial d_1}{\partial K} - e^{-RT}\Phi(d_2) - Ke^{-RT}\varphi(d_2)\frac{\partial d_2}{\partial K}$

$$= -e^{-RT}\Phi(d_2) - \frac{1}{\sigma\sqrt{T}}\left(\frac{S_0\varphi(d_1)}{K} - e^{-RT}\varphi(d_2)\right)$$

$$\left(\varphi(d_2) = e^{-\frac{d_2^2}{2}} = e^{-\frac{d_1^2}{2}} e^{\sigma\sqrt{T}d_1} e^{-\frac{\sigma^2 T}{2}} = \varphi(d_1)\, e^{\sigma\sqrt{T}d_1} e^{-\frac{\sigma^2 T}{2}}\right)$$

$$= -e^{-RT}\Phi(d_2) - \frac{\varphi(d_1)}{\sigma\sqrt{T}}\left(\frac{S_0}{K} - e^{-RT-\frac{\sigma^2 T}{2}} \underbrace{e^{\sigma\sqrt{T}d_1}}_{= e^{RT+\frac{\sigma^2 T}{2}}\frac{S_0}{K}}\right)$$

$$= -e^{-RT}\Phi(d_2) < 0 \quad \forall K.$$

Hence, V_0 is decreasing in K as expected.