On the Realizability of Electric Fields in Conducting Materials

Given an electric field $\nabla u$ in $\mathbb{R}^d$, we discuss whether it is realizable in the sense that there is a conductivity matrix function $\sigma$ such that the current field given by Ohm’s law, $\sigma \nabla u$, satisfies conductivity equation

$$\text{div}(\sigma \nabla u) = 0.$$ 

For application we focus on periodic fields. Isotropic conductivities given by a scalar $\sigma = fI$ may be found by solving a first order partial differential equation. A smooth, non-vanishing, electric field may be isotropically realized locally. If the flow of the electric field has a global section, which happens for periodic fields, then it may be realized globally, but the conductivity may not be periodic. If, additionally, its divergence is integrable along trajectories, it may be isotropically realized with a periodic conductivity. However, non-smooth electric fields may not be realizable. If a periodic field with nonzero cell average is realized, it must be non-vanishing pointwise in two dimensions but may vanish in three.

In the anisotropic case where electric fields are realized by a periodic symmetric positive definite matrix valued conductivities, we characterize the realizable periodic gradient fields in two dimensions. We consider also vector valued potentials and their matrix valued fields. Smooth periodic matrix fields are anisotropically realizable if they have positive determinants. Analogously, laminate electric fields are anisotropically realizable if all of their matrix fields have positive determinants. [v., “Which electric fields are realizable in conducting materials?,” ESAIM: Mathematical Modeling and Numerical Analysis 2013/89: Special Issue 2014: Multiscale Problems and Techniques, Archive http://arxiv.org/abs/1301.1613.]
On the Realizability of Electric Fields in Conducting Materials

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Tuesday, July 22, 2014
Manuscript is available

“Which electric fields are realizeable in conducting materials?”


http://arxiv.org/abs/1301.1613
3. Outline.

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      - $d = 2$ non-vanishing and $d = 3$ periodic chain mail
    - Anisotropic Realizability
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4. Conductivity $\Rightarrow$ Electric Field. Electric Field $\Rightarrow$ Conductivity?

- Let $Y = [0, 1]^d$ be the unit cube in $\mathbb{R}^d$ and $\sigma \in L^\infty(\mathbb{R}^d, \mathbb{R}^{d \times d})$ be symmetric, uniformly elliptic conductivity. Assume $\sigma$ is $Y$-periodic:

$$\sigma(x + k) = \sigma(x) \quad \text{for all } x \in \mathbb{R}^d \text{ and } k \in \mathbb{Z}^d.$$  

For all $\lambda \in \mathbb{R}^d - \{0\}$ there is $u^\lambda \in H^1_{\text{loc}}(\mathbb{R}^d)$, unique up to constant multiple, such that $u(x) - \lambda \cdot x$ is $Y$-periodic and

$$\text{div}(\sigma \nabla u^\lambda) = 0 \quad (1)$$

The effective conductivity of the periodic material is then $\sigma^*$ given by averaging over a cell

$$\sigma^* \lambda = \sigma^* \langle \nabla u^\lambda \rangle = \langle \sigma \nabla u^\lambda \rangle$$

where $\nabla u^\lambda$ is the electric field and $J = \sigma \nabla u^\lambda$ is the current field.

- We reverse the question: given a periodic electric field $\nabla u$, is it possible to find a symmetric periodic positive definite conductivity $\sigma$ that satisfies the conductivity equation (1)? In other words, which electric fields are realizable?
Consider the case that conductivity $\sigma = sI$ is isotropic.

**Theorem (II 1)**

*Assume that $u \in C^2(\mathbb{R}^d)$ satisfies $\nabla u \neq 0$. Then $\nabla u$ is locally isotropically realizable.*

Let $x_0 \in D$. Writing $\sigma = e^z$ the conductivity equation $\text{div}(e^z \nabla u) = 0$ becomes a first order PDE for the unknown $z(x)$,

$$\nabla u(x) \cdot \nabla z = -\Delta u(x) \quad (2)$$

The usual method of characteristics gives the solution. Since $\nabla u$ is a characteristic direction, if $\mathcal{H}$ is a hypersurface through $x_0$, transverse to $\nabla u(x_0)$ and $z_0(h)$ a function on $\mathcal{H}$, then the solution may be given by using the PDE to propagate the solution off of $\mathcal{H}$. Let $X(t, x)$ be the gradient flow of $\nabla u$, satisfying the characteristic ODE for $(t, h) \in I \times G$ in some neighborhood $G$ of $x_0$ and some $I = (-\varepsilon, \varepsilon)$ where $\varepsilon > 0$,

$$\frac{\partial}{\partial t} X(t, h) = \nabla u(X(t, h)), \quad \text{for } (t, h) \in I \times G$$

$$X(0, h) = h$$
Then $z$ satisfies an ODE along the trajectories since

$$\frac{\partial}{\partial t} z(X(t, h)) = \nabla z(X(t, h)) \cdot \frac{\partial}{\partial t} X(t, h) = -\Delta u(X(t, h))$$

If also the initial condition holds

$$z(0, h) = z_0(h) \quad \text{for } h \in G \cap H$$

then the solution is

$$\zeta(t, h) = z_0(h) - \int_0^t \Delta u(X(\tau, h)) \, d\tau$$

Finally, the mapping $\Psi : (t, h) \mapsto X(t, h)$ is a local $C^1$ diffeomorphism from $I \times (G \cap H)$ to a neighborhood of $x_0$ since the Jacobian $d\Psi(0, x_0)$ is invertible because $\nabla u(x_0)$ is transverse to $H$. Writing its inverse $(t, h) = \Phi(x)$, a solution of (2) near $x_0$ is

$$z(x) = \zeta(\Phi(x)).$$
We say that the hypersurface $\mathcal{H}$ is a **global section** for the flow of $\nabla u$ if the trajectory of the gradient flow starting from any point $y \in \mathbb{R}^d$ meets $\mathcal{H}$ transversally in exactly one point.

**Theorem (I 2)**

*Assume that $u \in C^2(\mathbb{R}^d)$ satisfies $\nabla u \neq 0$ and that $\nabla u$ has a global section $\mathcal{H}$. Then $\nabla u$ is isotropically globally realizable.*

Note that if $\nabla u$ is periodic then $z$ may not be periodic.
Example (1)

Let \( u(x, y) = x - \cos(2\pi y) \), and \( Y = [0, a] \times [0, 1] \), where \( a > 0 \).

\[
\nabla u = e_1 + 2\pi \sin(2\pi y)e_2, \quad \Delta u = 4\pi^2 \cos(2\pi y).
\]

On the section \( x = x_1 \) the initial condition is \( X(0, x) = x \) and the gradient flow decouples

\[
\frac{\partial}{\partial t} X_1 = 1 \\
\frac{\partial}{\partial t} X_2 = 2\pi \sin(2\pi X_2(t, x))
\]

It can be integrated: for \( x = (x_1, x_2) \)

\[
X(t, x) = \left( x_1 + t, n + \frac{1}{\pi} \arctan(e^{4\pi^2 t} \tan(\pi x_2)) \right), \quad \text{if } x_2 \in \left( n - \frac{1}{2}, n + \frac{1}{2} \right)
\]

\[
X(t, x) = \left( x_1 + t, n + \frac{1}{2} \right), \quad \text{if } x_2 = n + \frac{1}{2}
\]
Also

\[ \frac{\partial}{\partial t} z = \nabla z \cdot \frac{\partial}{\partial t} X = -\Delta u(X(t, x)) = \frac{4\pi^2 e^{8\pi^2 t} \tan^2 (\pi x_2) - 4\pi^2}{e^{8\pi^2 t} \tan^2 (\pi x_2) + 1} \]

If \( z \) vanishes at \( x_1 = 0 \), this can be integrated to yield

\[ \sigma = e^z = \begin{cases} 
\frac{1 + \tan^2 (\pi x_2)}{e^{4\pi^2 x_1} + e^{-4\pi^2 x_1} \tan^2 (\pi x_2)}, & \text{if } x_2 \not\in \frac{1}{2} + \mathbb{Z}; \\
e^{4\pi^2 x_1}, & \text{if } x_2 \in \frac{1}{2} + \mathbb{Z};
\end{cases} \]

We see it is not periodic in \( Y \).
Figure: Trajectories of the Gradient Flow for Example 1.
11. Local Isotropic Realizability May Fail for Discontinuous $\nabla u$.

Example (2)

Let the characteristic function of periodic intervals be given by
$\chi(t) = 1$ if $0 \leq \lfloor t \rfloor \leq \frac{1}{2}$ (fractional part) and 0 otherwise. Then

$$u(x, y) = y - x + \int_0^x \chi(t) \, dt$$

is Lipschitz continuous and

$$\nabla u = \chi e_2 + (1 - \chi)(e_2 - e_1) \quad \text{a.e. in } \mathbb{R}^2,$$

For this $\nabla u$ there is no positive function $\sigma \in L^\infty(\mathbb{R}^2)$ such that $\sigma \nabla u$ is divergence free.

$\nabla u$ has discontinuities on the lines $x_1 = k/2$ for some $k \in \mathbb{Z}$. Let $Q = (-r, r)^2$ for some $r \in (0, \frac{1}{2})$. If there were positive $\sigma \in L^\infty(Q)$ such that $\sigma \nabla u$ is divergence free, then there is a stream function $\nu \in H^1$ satisfying $\nabla \nu = R\sigma \nabla u$, which is unique up to additive constant and is Lipschitz continuous.
\[ \nabla v = R\sigma \nabla u \] implies

\[ 0 = \nabla u \cdot \nabla v = (e_2 - e_1) \cdot \nabla v \quad \text{in} \ (-r, 0) \times (-r, r) \]

hence \( v(x, y) = f(x + y) \) for some Lipschitz function \( f \) in \([-2r, r]\). On the other hand

\[ 0 = \nabla u \cdot \nabla v = e_2 \cdot \nabla v \quad \text{in} \ (0, r) \times (-r, r) \]

Hence \( v(x, y) = g(x) \) for some Lipschitz function \( g \) in \([0, r]\).

By continuity on the line \( x_1 = 0, f(y) = g(0) \). Hence \( f \) is constant on \([-r, r]\) implying \( v \) is too. Thus

\[ \nabla v = 0 \ a.e. \quad \text{in} \ (-r, 0) \times (0, r) \] and
\[ \sigma \nabla u = \sigma (e_2 - e_1) \neq 0 \ a.e. \quad \text{in} \ (-r, 0) \times (0, r) \]

which contradicts the equality \( \nabla v = R\sigma \nabla u \) a.e. Thus \( \nabla u \) is not isotropically realizable in neighborhoods near the lines \( x = k/2, k \in \mathbb{Z} \).
Let \( Y \in \mathbb{R}^d \) be a closed parallelepiped. Assume that \( u \in C^1(\mathbb{R}^d) \) satisfies

- \( \nabla u \) is \( Y \)-periodic and the cell average \( \langle \nabla u \rangle \neq 0 \).
- \( \nabla u \) is realized as an electric field associated with a smooth periodic conductivity.

Then

1. if \( d = 2 \) then \( \nabla u \neq 0 \) in all of \( \mathbb{R}^2 \);
2. if \( d = 3 \) then there is an example where \( \nabla u(y_0) = 0 \) for some point \( y_0 \in \mathbb{R}^3 \).

(2) One example is given by Ancona[2002], another may be constructed from the periodic chain mail of Briane, Milton and Nesi[2004].
Theorem (Alessandrini & Nesi (2001))

Let $Y \subset \mathbb{R}^2$ be a parallelogram, $\sigma \in \mathcal{L}^\infty$ be uniformly positive definite, symmetric and $Y$-periodic. For a symmetric matrix $A$ with $\det A > 0$ consider $U \in \mathcal{W}^{2,2}_{loc}(\omega, \mathbb{R}^2)$ such that $U - Ax$ is a $Y$-periodic and satisfies

$$\text{Div}(\sigma DU) = 0$$

and the cell average $\langle \det(DU) \rangle > 0$. Then

$$\det(DU) > 0 \quad \text{a.e. in } \mathbb{R}^2.$$ 

In the isotropic case $u$ is a scalar, $\langle \nabla u \rangle \neq 0$ implies $\nabla u \neq 0$ in $\mathbb{R}^2$. 
Figure: Periodic chain mail of Briane, Milton and Nesi consisting of linked toroidal rings of highly conductive material.
Rings have $\sigma \gg 1$. There is a matrix field such that $\langle DU \rangle = I$, $\langle \det(DU) \rangle = 1$ but $\det(DU) < 0$ in green region. Hence there is $\lambda \in \mathbb{R}^3 - \{0\}$ such that $\nabla (u \cdot \lambda)$ vanishes in $\mathbb{R}^3$. 

**Figure:** Section of periodic chain mail.
Theorem (I 4)

Let \( Y \subset \mathbb{R}^d \) be a compact paralleloiped and \( d \geq 2 \). Let \( u \in C^3(\mathbb{R}^d) \) such that \( \nabla u \) is \( Y \)-periodic,

\[
\nabla u \neq 0 \text{ in } \mathbb{R}^d \text{ and the cell average } \langle \nabla u \rangle \neq 0.
\]

Then \( \nabla u \) is globally isotropically realizable.

Since \( \nabla u \) is nonvanishing and periodic, \( 0 < c_1 \leq |\nabla u(x)| \leq c_2 \) for all \( x \) and the function \( f(t) = u(X(t, x_0)) \) satisfies

\[
f'(t) = \nabla u(X(t, x - 0)) \cdot \frac{\partial X}{\partial t}(t, x_0) = |\nabla u(X(t, x_0))|^2 \in [c_1^2, c_2^2].
\]

Thus

\[
\lim_{t \to \infty} f(t) = \infty \text{ and } \lim_{t \to -\infty} f(t) = -\infty
\]

and there is a unique \( \tau(x) \in \mathbb{R} \) such that \( f(\tau(x)) = 0 \). By differentiable dependence and the implicit function theorem \( \tau \in C^2(\mathbb{R}^d) \).
Hence the level set \( \{ x \in \mathbb{R}^d : \tau(x) = 0 \} \) is a \( C^1 \) global section. Put

\[
w(x) = \int_0^{\tau(x)} \Delta u(X(s, x)) \, ds \quad \text{for } x \in \mathbb{R}^d
\]

By the change of variables formula \( r = s + t \),

\[
w(X(t, x)) = \int_0^{\tau(x)-t} \Delta u(X(s + t, x)) \, ds = \int_t^{\tau(x)} \Delta u(X(r, x)) \, dr
\]

so

\[
\frac{\partial}{\partial t} w(X(t, x)) = \nabla w(X(t, x)) \cdot \nabla u(X(t, x)) = -\Delta u(X(t, x))
\]

For the conductivity \( \sigma = e^{w(x)} \) we have at \( t = 0 \)

\[
\text{div} (\sigma \nabla (u)) = e^w (\nabla w \cdot \nabla v + \Delta u) = 0.
\]
Theorem (I5. For Example 1, no periodic isotropic $\sigma$ is possible.)

For $u(x, y) = x - \cos(2\pi y)$, the $Y = [0, a] \times [0, 1]$-periodic electric field $\nabla u$ does not admit a continuous non-vanishing $Y$-periodic isotropic conductivity $\sigma$ that makes $\sigma \nabla u$ divergence free.

Note $\nabla u = e_1 + 2\pi \sin(2\pi y)e_2$. Assume there is a $Y$-periodic function $\sigma$ such that $\sigma \nabla u$ is divergence free. Let $Q = [0, a] \times [-r, r]$ for some $0 < r < \frac{1}{2}$. Then, using Green’s Theorem,

$$0 = \int_Q \text{div}(\sigma \nabla u) \, dx \, dy = \oint_{\partial Q} (\sigma u_x) \, dy - (\sigma u_y) \, dx$$

$$= \int_{-r}^{r} \left[ \sigma(a, y)u_x(a, y) - \sigma(0, y)u_x(0, y) \right] \, dy$$

$$+ \int_{0}^{a} \left[ \sigma(x, r)u_y(x, r) - \sigma(x, -r)u_y(x, -r) \right] \, dx$$

$$= 0 + 2\pi \sin(2\pi r) \int_{0}^{a} \left[ \sigma(x, r) + \sigma(x, -r) \right] \, dx > 0 \quad \square$$
Theorem (II 6.)

Let $Y \subset \mathbb{R}^d$ be a compact parallelopiped and $d \geq 2$. Let $u \in C^3(\mathbb{R}^d)$ such that $\nabla u$ is $Y$-periodic, $\nabla u \neq 0$ in $\mathbb{R}^d$ and the cell average $\langle \nabla u \rangle \neq 0$. Assume that there is $C < \infty$ such that for all $x \in \mathbb{R}^d$,

$$\left| \int_0^{\tau(x)} \Delta u(X(t, x)) \, dt \right| \leq C$$

(4)

where $\tau(x)$ is the unique time such that $u(\tau(x), x) = 0$ as in the proof of Theorem II 4. Then $\nabla u$ is isotropically realizable with $Y$-periodic conductivity $\sigma, \sigma^{-1} \in \mathcal{L}_Y^\infty(\mathbb{R}^d)$.

Conversely, if $\nabla u$ is isotropically realizable with $Y$-periodic conductivity $\sigma \in C^1_Y(\mathbb{R}^d)$, then (4) holds.
Example (1, cont. Assumptions of Theorem II.6 do not hold.)

Let \( u(x, y) = x - \cos(2\pi y) \), and \( Y = [0, a] \times [0, 1] \), where \( a > 0 \).

\[
\nabla u = (1, 2\pi \sin(2\pi x_2)) \\
\Delta u = 4\pi^2 \cos(2\pi x_2)
\]

Put \( p_0 = (x_1, 0) \). Thus, \( X(t, p_0) = (x_1 + t, 0) \) so

\[
w(p_0) = \int_0^{\tau(p_0)} \Delta u(X(t, p_0)) \, dt = 4\pi^2 \cos(0)\tau(p_0).
\]

But by the definition of \( \tau \),

\[
0 = u(X(\tau(p_0), p_0)) = x_1 + \tau(p_0) - \cos(0)
\]

so that

\[
w(p_0) = 4\pi^2(1 - x_1)
\]

which is not bounded.
Proof of Theorem II 6 (Necessity).

Assume there is a positive periodic $\sigma = e^w \in C^1_Y(\mathbb{R}^d)$ such that $\text{div}(\sigma \nabla u) = 0$. Then $\nabla u \cdot \nabla w + \Delta u = 0$ in $\mathbb{R}^d$. Hence

$$\int_0^{\tau(x)} \Delta u(X(t, x)) \, dt = - \int_0^{\tau(x)} \nabla w(X(t, x)) \cdot \nabla u(X(t, x)) \, dt$$

$$= - \int_0^{\tau(x)} \nabla w(X(t, x)) \cdot \frac{\partial}{\partial t} X(t, x) \, dt$$

$$= w(X(0, x)) - w(X(\tau(x), x))$$

$$= x - w(X(\tau(x), x))$$

which is bounded by assumption. Hence (4) follows.
For simplicity, assume $Y = [0, 1]^d$. For $x \in \mathbb{R}^d$ define

$$
\sigma_0(x) = \exp \left( \int_0^{\tau(x)} \Delta u(X(t, x)) \, dt \right)
$$

and for $n \in \mathbb{N}$, average over the $(2n + 1)^d$ integer vectors in $[-n, n]^d$

$$
\sigma_n(x) = \frac{1}{(2n + 1)^d} \sum_{k \in \mathbb{Z}^d \cap [-n, n]^d} \sigma_0(x + k)
$$

By (4), $\sigma_n$ is bounded in $\mathcal{L}^\infty(\mathbb{R}^d)$. Hence a subsequence $\sigma_{n'}$ converges weak-$*$ to $\sigma_\infty$ in $\mathcal{L}^\infty(\mathbb{R}^d)$. 
For any \( k \in \mathbb{Z}^d \)

\[
\left| (2n + 1)^d \sigma_n(x + k) - (2n + 1)^d \sigma_n(x) \right|
= \left| \sum_{|j - k|_\infty \leq n} \sigma_n(x + j) - \sum_{|j|_\infty \leq n} \sigma_n(x + j) \right|
\leq \sum_{|j|_\infty \leq n + |k|_\infty, |j|_\infty > n} \sigma_n(x + k) + \sum_{|j - k|_\infty \leq n + |k|_\infty, |j - k|_\infty > n} \sigma_n(x + k)
\leq 2e^C \left( (2n + 2k + 1)^d - (2n + 1)^d \right) \leq C_2(C, d, k)n^{d-1}
\]

Letting \( n' \to \infty \) implies that \( \sigma_\infty(x + k) = \sigma_\infty(x) \) a.e. in \( \mathbb{R}^d \) and for any \( k \). Thus \( \sigma_\infty \in L_\infty^Y(\mathbb{R}^d) \). Since \( \sigma_0 \) is bounded below by \( e^{-C} \), \( \sigma_\infty^{-1} \in L_\infty^Y(\mathbb{R}^d) \).
As $\nabla u \in C^2_Y(\mathbb{R}^d)$, it is realized by the conductivity $\sigma_0$. Periodicity implies that also $\text{div}(\sigma_n \nabla u) = 0$ in $\mathbb{R}^d$. From weak-$\ast$ convergence, for every $\varphi \in C_c^\infty(\mathbb{R}^d)$ we have

$$0 = \lim_{n' \to \infty} \int_{\mathbb{R}^d} \sigma_{n'} \nabla u \cdot \nabla \varphi \, dx = \int_{\mathbb{R}^d} \sigma_\infty \nabla u \cdot \nabla \varphi \, dx$$

Hence $\text{div}(\sigma_\infty \nabla u) = 0$ in $\mathcal{D}'(\mathbb{R}^d)$ so that $\nabla u$ is isotropically realized by the $Y$-periodic conductivity $\sigma_\infty$. \qed
26. Anisotropic Realizability.

**Theorem (A 1)**

Let \( Y \subset \mathbb{R}^2 \) be a closed parallelogram. Let \( u \in \mathcal{C}^1(\mathbb{R}^2) \) such that \( \nabla u \neq 0 \) is \( Y \)-periodic in \( \mathbb{R}^2 \) and the cell average \( \langle \nabla u \rangle \neq 0 \). Then necessary and sufficient that \( \nabla u \) be realizable by a continuous, \( Y \)-periodic, symmetric positive definite matrix-valued conductivity \( \sigma \) is that there is a function \( v \in \mathcal{C}^1(\mathbb{R}^2) \) such that \( \nabla v \) is \( Y \)-periodic in \( \mathbb{R}^2 \) and the cell average \( \langle \nabla v \rangle \neq 0 \) such that

\[
R \nabla u \cdot \nabla v = \det(\nabla u, \nabla v) > 0 \quad \text{everywhere in } \mathbb{R}^2. \tag{5}
\]

where \( R \) is rotation by a right angle.

Theorem A 1 continues to hold under the weaker assumptions that \( \nabla u \) is \( Y \)-periodic, \( \nabla u \in \mathcal{L}^2(Y) \), \( \nabla u \neq 0 \) a.e. in \( \mathbb{R}^2 \) and \( \langle \nabla u \rangle \neq 0 \). In this case, the \( Y \)-periodic conductivity \( \sigma \) defined only a.e. by the formula below and does not remain bounded in \( Y \). However \( \sigma \nabla u \) is divergence free in the sense of distributions.
Assume there is such \( v \). (5) says that \( \nabla v \) is nonvanishing. Then define

\[
\sigma = \frac{1}{|\nabla u|^4} \begin{pmatrix}
\frac{\partial u}{\partial x_1} & \frac{\partial u}{\partial x_2} \\
-\frac{\partial u}{\partial x_2} & \frac{\partial u}{\partial x_1}
\end{pmatrix}^T \begin{pmatrix}
R \nabla u \cdot \nabla v & -\nabla u \cdot \nabla v \\
-\nabla u \cdot \nabla v & \frac{|\nabla u \cdot \nabla v|^2 + 1}{R \nabla u \cdot \nabla v}
\end{pmatrix} \begin{pmatrix}
\frac{\partial u}{\partial x_1} & \frac{\partial u}{\partial x_2} \\
-\frac{\partial u}{\partial x_2} & \frac{\partial u}{\partial x_1}
\end{pmatrix}
\]

which is a continuous, symmetric positive definite matrix function.

\( \sigma \nabla u = -R \nabla v \) in \( \mathbb{R}^2 \) so it is divergence free. Now assume there is \( u \) and a continuous positive definite symmetric \( \sigma \).

Let \( v \in C^1(\mathbb{R}^2) \) be the stream function which satisfies \( \nabla v = -R \nabla u \).

Hence \( \nabla v \) is \( \gamma \)-periodic and

\[
R \nabla u \cdot \nabla v = -\nabla u \cdot R \nabla v = \sigma \nabla u \cdot \nabla u
\]

Allesandrini & Nesi's result implies \( \nabla u \) is nonvanishing, which implies (5). By the div-curl lemma,

\[
\langle R \nabla u \cdot \nabla v \rangle = R \langle \nabla u \rangle \cdot \langle \nabla v \rangle = \langle \sigma \nabla u \cdot \nabla u \rangle > 0
\]

so \( \langle \nabla v \rangle > 0 \) also.
Example (1)

Let \( u(x, y) = x - \cos(2\pi y) \), and \( Y = [0, a] \times [0, 1] \), where \( a > 0 \). Then \( \nabla u \) is anisotropically realizable.

\[
\nabla u = e_1 + 2\pi \sin(2\pi x_2) e_2.
\]

Take

\[
v(x) = x_2
\]

We find

\[
R \nabla u \cdot v = (-2\pi \sin(2\pi x_2) e_1 + e_2) \cdot e_2 = 1
\]

so Theorem A.1 applies: for \( \delta = 1 + 4\pi^2 \sin^2(2\pi x_2) \), let

\[
\sigma = \frac{1}{\delta^2} \begin{pmatrix}
\delta^2 + \delta - 1 & -2\pi \sin(2\pi x_2) \\
-2\pi \sin(2\pi x_2) & 1
\end{pmatrix}
\]

Now \( \sigma \nabla u = e_1 \) which is divergence free.
Example (2)

\[ u(x) = x_2 - x_1 + \int_0^{x_1} \chi(t) \, dt \] where \( \chi = 1 \) if \( 0 \leq |t| \leq \frac{1}{2} \) and 0 otherwise.

Then \( \nabla u \) is anisotropically realizable.

\[ \nabla u = \chi e_2 + (1 - \chi)(e_2 - e_1) \] a.e. in \( \mathbb{R}^2 \) satisfies the weaker assumptions. For a.e. \( x \in \mathbb{R}^2 \), define

\[ v(x) = -x_2 - \int_0^{x_1} \chi(t) \, dt, \quad \nabla v = -\chi(e_1 + e_2) - (1 - \chi)e_2 \]

so that a.e. in \( \mathbb{R}^2 \), \(-\nabla u \cdot \nabla v = R \nabla u \cdot \nabla v = 1\).

Then formula (5) yields the rank one laminate conductivity a.e. in \( \mathbb{R}^2 \),

\[ \sigma = \chi \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} + \frac{1 - \chi}{4} \begin{pmatrix} 5 & 1 \\ 1 & 1 \end{pmatrix} \]

Hence a.e. in \( \mathbb{R}^2 \), \( \sigma \nabla u = \chi(-e_1 + e_2) + (1 - \chi)e_2 \) which is divergence free in \( \mathcal{D}'(\mathbb{R}^2) \).
Let \( d \geq 2 \) and \( \Omega \subset \mathbb{R}^d \) be open. If \( U \in H^1(\Omega, \mathbb{R}^d) \) then the matrix electric field \( DU \) is said to be realizable if there is a symmetric positive definite matrix-valued function \( \sigma \in L^\infty_{\text{loc}}(\Omega, \mathbb{R}^{d \times d}) \) such that

\[
\text{Div}(\sigma DU) = 0
\]

**Theorem (M1)**

Let \( d \geq 2 \) and \( Y \subset \mathbb{R}^d \) be a closed parallelepiped. Let \( U \in C^1(\mathbb{R}^d, \mathbb{R}^d) \) such that \( DU \) is \( Y \)-periodic.

1. Assume also \( \det(\langle DU \rangle DU) > 0 \) in \( \mathbb{R}^d \) and \( \det(\langle DU \rangle) \neq 0 \). Then \( DU \) is a realizable matrix electric field with continuous conductivity.
2. If \( d = 2 \), \( \det(\langle DU \rangle) \neq 0 \) and the matrix electric field realized by a \( C^1 \) conductivity, then \( \det(\langle DU \rangle DU) > 0 \).
3. If \( d = 3 \) there exists a smooth \( Y \)-periodic matrix field \( DU \) such that \( \det(\langle DU \rangle) \neq 0 \) and an associated smooth periodic conductivity \( \sigma \) such that \( \det(DU) \) takes both positive and negative values in \( \mathbb{R}^3 \).
(i.) For $Y$-periodic $U \in C^1$ such that $\det(\langle DU \rangle DU) \neq 0$ we define

$$
\sigma = \det(\langle DU \rangle DU) (DU^{-1})^T DU^{-1} = \det(\langle DU \rangle) \text{Cof}(DU) DU^{-1}
$$

where Cof is the cofactor matrix. $\sigma$ is $Y$-periodic, continuous, symmetric and positive definite. Also by Piola’s identity, as a distribution,

$$\text{Div}(\text{Cof } DU) = 1 \quad \text{in } \mathcal{D}'(\mathbb{R}^d)$$

Hence $\sigma DU$ is divergence free and $DU$ is realizable with associated conductivity $\sigma$.

(ii.) Follows from a theorem of Alessandrini and Nesi.

(iii.) Example is constructed from periodic chain mail constructed by Briane, Milton and Nesi.
The result (i.) may be generalized:

**Corollary (M 2)**

Let \( d \geq 2 \) and \( Y \subset \mathbb{R}^2 \) be a closed parallelepiped. Let \( U \in C^1(\mathbb{R}^d, \mathbb{R}^d) \) with \( Y \)-periodic \( DU \), \( \det(\langle DU \rangle DU) > 0 \) in \( \mathbb{R}^d \) and \( \det(\langle DU \rangle) \neq 0 \). Then the matrix electric field \( DU \) is realized by a family of continuous conductivities \( \sigma_\varphi \) parameterized by convex functions \( \varphi \in C^2(\mathbb{R}^d) \), those whose Hessian matrices \( D^2 \varphi \) are positive definite everywhere in \( \mathbb{R}^d \).

Define

\[
\sigma_\varphi = \det(\langle DU \rangle) \ Cof(D(\nabla \varphi \circ U) DU^{-1})
\]

\( \sigma_\varphi DU \) is divergence free by Piola's identity. We also have

\[
Cof(D(\nabla \varphi \circ U)) = Cof(DU D^2 \varphi \circ U) = Cof(DU) \ Cof(D^2 \varphi \circ U)
\]

so that \( \sigma_\varphi \) satisfies

\[
\sigma_\varphi = \det(\langle DU \rangle DU) (DU^{-1})^T \ Cof(D^2 \varphi \circ U) DU^{-1}
\]

Since \( D^2 \varphi \) is symmetric positive definite, so is its cofactor matrix. Thus \( \sigma_\varphi \) is an admissible, continuous with \( \sigma_\varphi DU \) divergence free in \( \mathbb{R}^d \). \( \square \)
Figure: A rank-two laminate with directions $\xi_1 = e_1$ and $\xi_{1,2} = e_2$.

Let $d, n \in \mathbb{N}$. A rank-$n$ laminate in $\mathbb{R}^d$ is a multiscale microstructure defined at $n$ ordered scales $\varepsilon_n \ll \cdots \ll \varepsilon_1$ depending on a small positive parameter $\varepsilon \to 0$ and in multiple directions in $\mathbb{R}^d \setminus \{0\}$, by the following process.
At the smallest scale $\varepsilon_n$ there is a set of $m_n$ rank-one laminates, the $i$th one of which, for $i = 1, \ldots, m_n$, is composed of an $\varepsilon_n$ periodic repetition in the $\xi_{i,n}$ direction of homogeneous layers with constant positive definite conductivity matrices $\sigma_{i,n}^h$, $h \in I_{i,n}$.

At the scale $\varepsilon_k$ there is a set of $m_k$ laminates, the $i$th one of which, for $i = 1, \ldots, m_k$, is composed of an $\varepsilon_k$-periodic repetition in the $\xi_{i,k} \in \mathbb{R}^d \setminus \{0\}$ direction of homogeneous layers and/or a selection of the $m_{k+1}$ laminates which are obtained at stage $k + 1$ with constant positive definite conductivity matrices $\sigma_{i,k}^h$ and/or $\sigma_{i,j}^h$, resp., for $j = k + 1, \ldots, n$, $h \in I_{i,j}$.

At the scale $\varepsilon_1$ there is a single laminate ($m_1 = 1$) which is composed of an $\varepsilon_1$-periodic repetition in the $\xi_1 \in \mathbb{R}^d \setminus \{0\}$ direction of homogeneous layers and/or a selection of the $m_2$ laminates which are obtained at scale $\varepsilon_2$ with constant positive definite conductivity matrices $\sigma_{i,1}^h$ and/or $\sigma_{i,j}^h$, resp., for $j = 2, \ldots, n$, $h \in I_{i,j}$.

The laminate conductivity at stage $k = 1, \ldots, n$ is denoted by $L_k^{\varepsilon}(\hat{\sigma})$ where $\hat{\sigma}$ is the whole set of constant laminate conductivities.
Briane and Milton showed that there is a set $\hat{P}$ of constant $d \times d$ matrices such that $P_\varepsilon = L_n^\varepsilon(\hat{P})$ is a corrector (or a matrix electric field) associated to the conductivity $\sigma_\varepsilon = L_n^\varepsilon(\hat{\sigma})$ in the sense of Murat-Tartar:

\[
\begin{align*}
&\begin{cases}
P_\varepsilon \rightharpoonup I & \text{weakly in } \mathcal{L}_{loc}^2(\mathbb{R}^d, \mathbb{R}^{d \times d}), \\
\text{Curl}(P_\varepsilon) \to 0 & \text{strongly in } \mathcal{H}_{loc}^{-1}(\mathbb{R}^d, \mathbb{R}^{d \times d}), \\
\text{Div}(\sigma_\varepsilon P_\varepsilon) & \text{is compact in } \mathcal{H}_{loc}^{-1}(\mathbb{R}^d, \mathbb{R}^d).
\end{cases}
\end{align*}
\]

(6)

The weak limit of $\sigma_\varepsilon P_\varepsilon$ in $\mathcal{L}_{loc}^2(\mathbb{R}^d, \mathbb{R}^{d \times d})$ is then the homogenized limiting conductivity of the laminate. The three conditions (6) satisfied by $P_\varepsilon$ extend in the laminate case to the three respective conditions

\[
\begin{align*}
&\begin{cases}
\langle DU \rangle = I, \\
\text{Curl}(DU) = 0, \\
\text{Div}(\sigma DU) = 0.
\end{cases}
\end{align*}
\]

(7)

satisfied by any electric field $DU$ in the periodic case.
Theorem (L 1)

Let \(d, n \in \mathbb{N}\). Consider the rank-\(n\) laminate multiscale field \(L^n_\varepsilon(\hat{P})\) built from a finite set \(\hat{P}\) of \(d \times d\) matrices satisfying

\[
P_\varepsilon \rightharpoonup I \quad \text{weakly in } L^2_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^{d \times d}),
\]

\[
\text{Curl}(P_\varepsilon) \to 0 \quad \text{strongly in } H^{-1}_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^{d \times d}),
\]

Then necessary and sufficient that the field be realized, i.e., \(\text{Div}(\sigma_\varepsilon P_\varepsilon)\) is compact in \(H^{-1}_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^d)\) for some rank-\(n\) laminate conductivity \(L^n_\varepsilon(\hat{\sigma})\) is that \(\det(L^n_\varepsilon(\hat{P})) > 0\) a.e. in \(\mathbb{R}^d\), or equivalently, that the determinant of each matrix in \(\hat{P}\) is positive.

Determinant positivity follows from a theorem of Briane, Milton and Nesi. Conversely, suppose there is a laminate field \(P_\varepsilon = L^n_\varepsilon(\hat{P})\) satisfying (8) and \(\det(P_\varepsilon) > 0\) a.e.
As in the matrix field case consider the rank-$n$ conductivity defined by

$$\sigma_\varepsilon = \det(P_\varepsilon) (P_\varepsilon^{-1})^T P_\varepsilon^{-1} = L_\varepsilon^\ell(\hat{\sigma}),$$

where $\hat{\sigma} = \{\det(P) (P^{-1})^T P^{-1} : P \in \hat{P}\}$. Then compactness is equivalent to the compactness of

$$\text{Div}(\text{Cof}(P_\varepsilon)).$$

Contrary to the periodic case, Cof($P_\varepsilon$) is not divergence free as a distribution. But using the homogenization procedure for laminates of Briane, by the quasi-affinity of cofactors for gradients compactness holds if the matrices $P$ and $Q$ of two neighboring layers in a direction $\xi$ of the laminate satisfy the jump condition for the divergence

$$\left(\text{Cof}(P) - \text{Cof}(Q)\right)^T \xi = 0. \quad (9)$$
More precisely, at the given scale $\varepsilon_k$ of the laminate the matrix $P$ or $Q$ is either a matrix in $\hat{P}$ or the average of rank-one laminates obtained at the smallest scales $\varepsilon_{k+1}, \ldots, \varepsilon_n$

In the first case the matrix $P$ is the constant value of the field in a homogeneous layer of the rank-$n$ laminate.

In the second case, the average of the cofactors of the matrices involved in those rank one laminations is equal to the cofactors matrix of the average, $\text{Cof}(P)$, by virtue of the quasi-affinity of the cofactors applied iteratively to the rank-one connected matrices in the rank-one laminate.

Therefore, it remains to prove (9) for any matrices $P$ and $Q$ with positive determinant satisfying the condition that controls the jumps in the convergence of $\text{Curl}(P_{\varepsilon}) \to 0$ in (8).
For any matrices $P$ and $Q$ with positive determinant satisfying the condition we must show

$$P - Q = \xi \otimes \eta, \quad \text{for some } \eta \in \mathbb{R}^d.$$

So by multiplicativity of cofactor matrices we have

$$(\text{Cof}(P) - \text{Cof}(Q))^T \xi = \text{Cof}(Q)^T \left[ \text{Cof}(I + (\xi \otimes \eta)Q^{-1})^T - I \right]$$

$$= \text{Cof}(Q)^T \left[ \text{Cof}(I + \xi \otimes \lambda)^T - I \right]$$

where $\lambda = (Q^{-1})^T \eta$. Moreover if $\xi \cdot \eta \neq -1$ we have

$$\text{Cof}(I + \xi \otimes \lambda)^T = \det(I + \xi \otimes \lambda)(I + \xi \otimes \lambda)^{-1} = (I + \xi \cdot \lambda)I - \xi \otimes \lambda,$$

which extends to the case $\xi \cdot \eta = -1$ by continuity. Hence

$$(\text{Cof}(P) - \text{Cof}(Q))^T = \text{Cof}(Q)((\xi \cdot \lambda)I - \xi \otimes \lambda),$$

which implies (9) since $(\xi \otimes \lambda)\xi = (\xi \cdot \lambda)\xi$. \square
Thanks!