Approximating Roots to Polynomials of Several Complex Variables

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The Problem:
Given n polynomials in n complex variables with complex coefficients, we aim to solve the system

$$\begin{aligned} G_1(z_1, \ldots, z_n) &= 0 \\ &\vdots \\ G_n(z_1, \ldots, z_n) &= 0 \end{aligned}$$

using a **"homotopy"** from a similar system

$$\begin{aligned} K_1(z_1, \ldots, z_n) &= 0 \\ &\vdots \\ K_n(z_1, \ldots, z_n) &= 0 \end{aligned}$$

where the solution is known.

Homotopy Construction:
We create a time parametrized family of polynomials, $\{P_i\}$, given by

$$\begin{aligned} P_1(z_1, \ldots, z_n; t) &= (1-t)\, K_1 + t\, G_1 \\ &\vdots \\ P_n(z_1, \ldots, z_n; t) &= (1-t)\, K_n + t\, G_n \end{aligned}$$

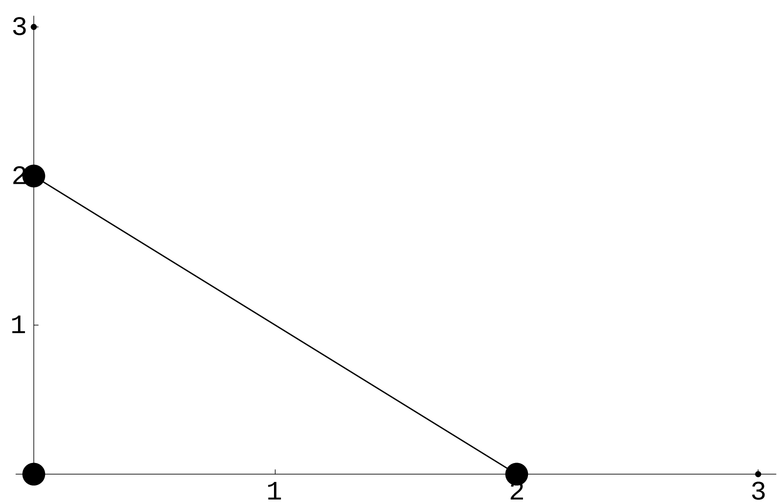
where $0 \le t \le 1$.

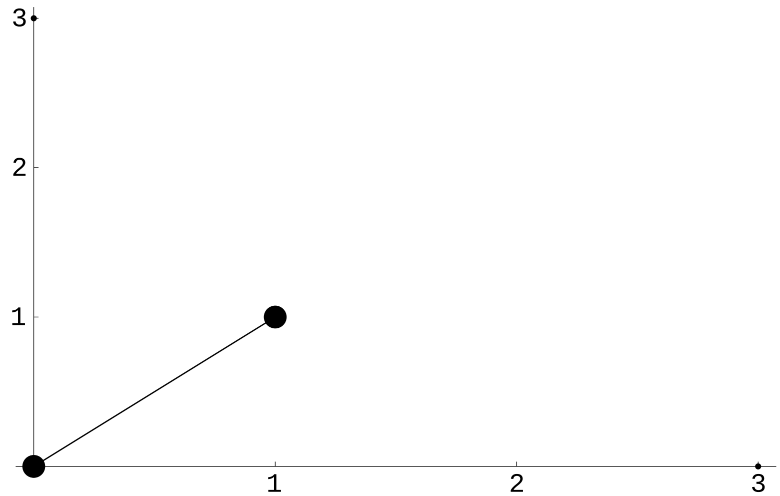
Finding the Known System;
Newton Polyhedra:
Consider one of the given polynomials, $G_i(z_1, \ldots, z_n)$. Take the individual terms, $a_{i,j}(z_1)^{i_1} \cdots (z_n)^{i_n}$, and create the lattice points $(i_1, \ldots, i_n)$ which depend upon the exponents of the variables. The convex hull of these points define a Newton polyhedron in dimension n.

For example, the Newton polyhedra for the system,

$$\begin{aligned} G_1(z_1, z_2) &= (z_1)^2 + (z_2)^2 - 1 \\ G_2(z_1, z_2) &= (z_1)(z_2) - 1 \end{aligned}$$

is shown below.





A Nice Result for $n = 2$:
From the given system, $\{G_1, G_2\}$, construct the Newton polyhedra, $\{C_1, C_2\}$. Then,

$$\#\text{ of roots } \leq A(C_1 + C_2) - A(C_1) - A(C_2) = m$$

where A is the area of the polyhedra, and $C_1 + C_2$ is the Minkowski sum. The known system becomes,

$$\begin{aligned} K_1(z_1, z_2) &= (z_1)^m - 1 \\ K_2(z_1, z_2) &= (z_2)^m - 1 \end{aligned}$$

which has the **"roots of unity"** as the initial roots,

$$e^{\frac{2k\pi i}{m}}$$

with $0 \leq k \leq m - 1$.

Evaluation Procedure:

1. Begin at $t = 0$ and store the initial roots from the known system in some set, S.

2. Increment time by Δt and apply Newton's method to the family, $\{P_i\}$, using the previous values from S.

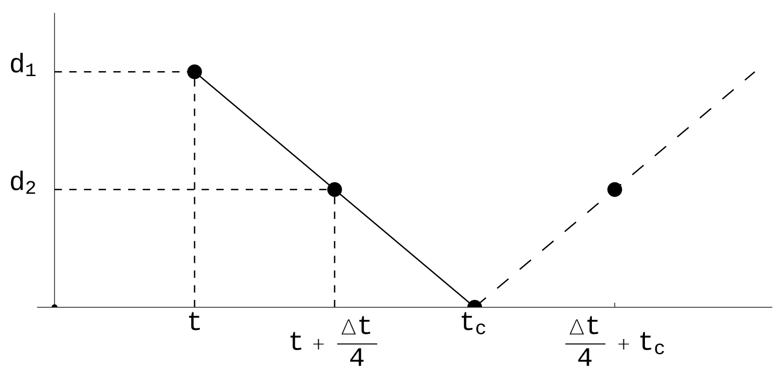
3. Update the current values in S using the values from step 2.
4. Repeat steps 2 and 3 unitl $t = 1$.

Complications:

1. Root collisions: if at any time during the evaluation procedure, two or more roots in the set S get **"too close"** to each other, Newton's method $\to \infty$.
2. Too many initial roots: depending upon which system one begins with, there may be too many initial roots at $t = 0$.

Resolution (Collisions):
Under the homotopy map, colliding roots will converge to each other, rotate by 90°, and then diverge from each other.

1. Create a **"warning distance"** parameter which halts the evaluation.
2. Adjust the evaluation procedure and predict the moment of collision, t_c.
   - Back up one time step ($t \to t - \Delta t$), and compute a few more Newton steps with time fixed.
   - Find the squared distance between the colliding roots, d_1.
   - Incriment time by $t \to t + \frac{\Delta t}{4}$, compute a few more Newton steps with time fixed, and find another distance between the roots, d_2.



- By similar triangles,

$$t_c = \frac{(t + \Delta t/4)d_2}{d_1 - d_2}$$

- Multiply the colliding roots by $\pm i$ and replace the new roots into S.

3. Continue the loop with $t \to t_c + \frac{\Delta t}{4}$ until another collision is detected.

Resolution (Too Many Initial Roots):
Most often, if one starts with the **"roots of unity"**, there will be more initial roots than actual roots. This is due to the inequality in the Minkowski formula. However, these **"fake"** roots **have yet** to cause problems in the evaluation, as the distances between them and the actual roots becomes large.

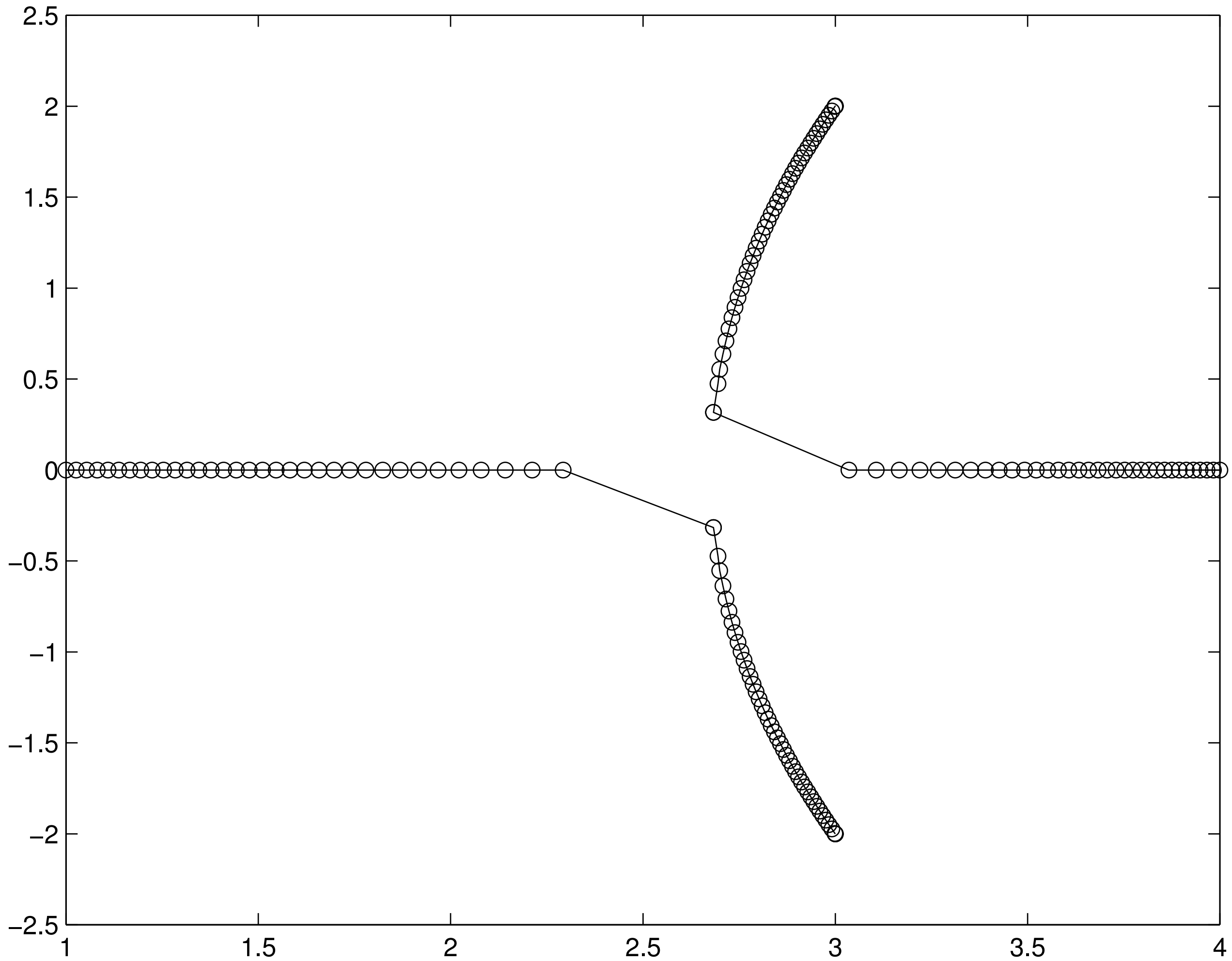
Example #1 (Illustration of Root Collision):
Consider the given polynomial in one real variable,

$$G(x) = 13 - 6x + x^2$$

We use,

$$K(x) = 4 - 5x + x^2$$

as the known polynomial with roots $\{1, 4\}$. The solution to $G(x) = 0$ is $x = 3 \pm 2i$.



Example #2
Consider the given system,

$$\begin{aligned} G_1(z_1, z_2) &= (z_1)^2 + (z_2)^2 - 1 \\ G_2(z_1, z_2) &= (z_1)(z_2) - 1 \end{aligned}$$

The maximum number of roots is,

$$\begin{aligned} \#\text{ of roots} &\leq A(C_1 + C_2) - A(C_1) - A(C_2) \\ &= 6 - 2 - 0 \end{aligned}$$

The known system becomes,

$$\begin{aligned} K_1(z_1, z_2) &= (z_1)^4 - 1 \\ K_2(z_1, z_2) &= (z_2)^4 - 1 \end{aligned}$$

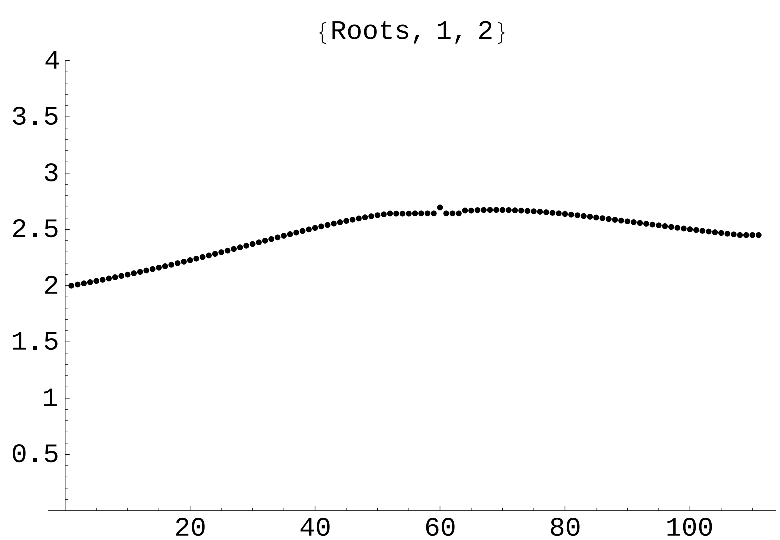
Initial roots that converged to the actual roots:

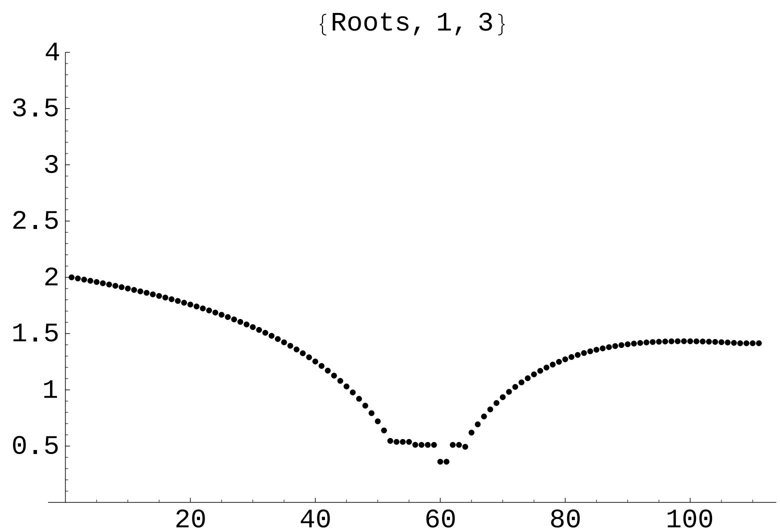
$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

The actual roots to the given system:

$$\begin{pmatrix} 0.866 + 0.5i \\ 0.866 - 0.5i \end{pmatrix} \quad \begin{pmatrix} -0.866 + 0.5i \\ -0.866 - 0.5i \end{pmatrix}$$

$$\begin{pmatrix} 0.866 - 0.5i \\ 0.866 + 0.5i \end{pmatrix} \quad \begin{pmatrix} -0.866 - 0.5i \\ -0.866 + 0.5i \end{pmatrix}$$

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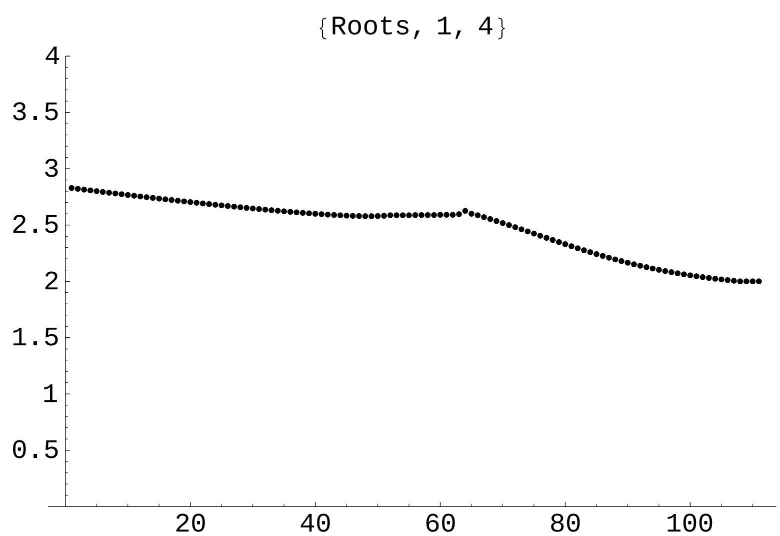
4
3.5
3
2.5
2
1.5
1
0.5
20
40
60
80
100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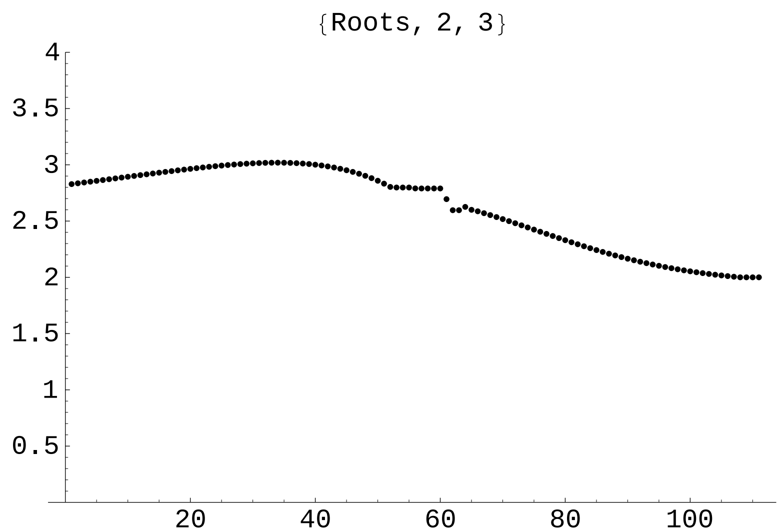
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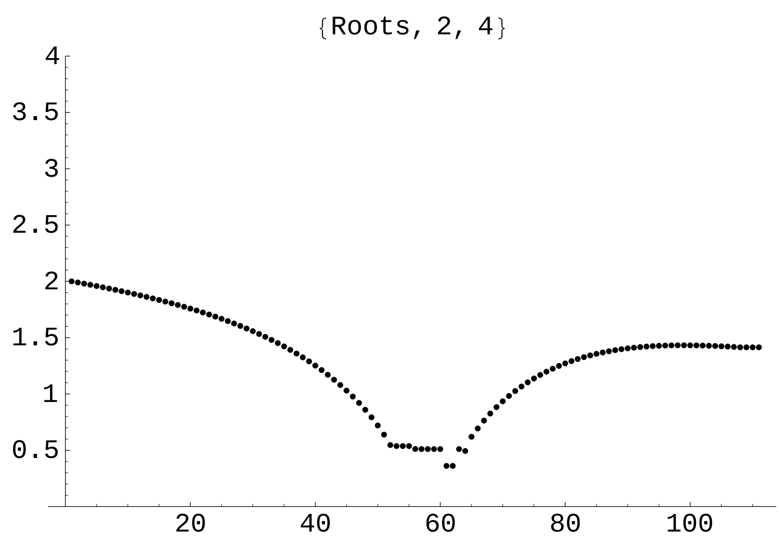
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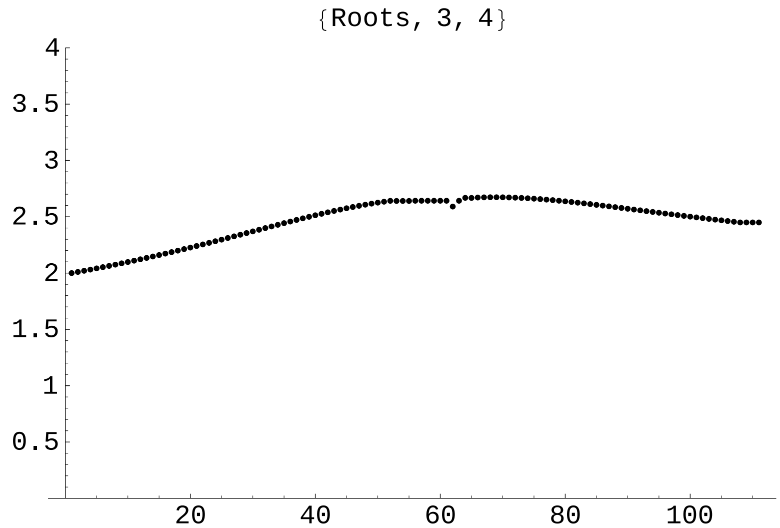
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