

UNITARY FUNCTORIAL CORRESPONDENCES FOR p -ADIC GROUPS

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ABSTRACT. In this paper, we generalize the results of [BM1, BM2] to affine Hecke algebras of arbitrary isogeny class with geometric unequal parameters, and extended by groups of automorphisms of the root datum. When the theory of types ([BK1, BK2]) gives a Hecke algebra of the form considered in this paper, our results establish a transfer of unitarity from the corresponding Bernstein component of the category of smooth representations of p -adic groups to the associated categories of Hecke algebra modules, as well as unitary functorial correspondences between certain Bernstein components of possibly different p -adic groups.

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1. INTRODUCTION

1.1. A central result in the representation theory of p -adic groups, is the Borel-Casselman equivalence between the categories of representations with Iwahori fixed vectors, and finite dimensional modules of the affine Hecke algebra, [Bo1]. These results have served as a paradigm for subsequent efforts to solve the problem of classifying the admissible dual of a p -adic group, [BK1, BK2], [Ro], [Se] and many others. From the outset it was conjectured that the Borel-Casselman correspondence also preserves unitarity. In [BM1], it is proved that, for a split group G of adjoint type over a p -adic field \mathbb{F} , the Borel-Casselman equivalence between the categories of representations with Iwahori fixed vectors, and finite dimensional modules of the affine Hecke algebra preserves unitarity. The results depend heavily on the fact that the affine Hecke algebras has equal parameters, and corresponds to an adjoint group. In addition, one has to make certain technical assumptions, namely that the infinitesimal character be real (*cf.* 1.3). This assumption is removed in [BM2].

The first purpose of this paper is to establish the preservation of unitarity in a much more general setting. To do this, we treat the case of an extended affine Hecke algebra with general parameters of geometric type, and coming from a group of arbitrary isogeny class. The work of [BM1] relies on combining the classification of representations of the affine Hecke

algebra and the idea of signature character in [Vo]. In order to treat the larger class of Hecke algebras in this paper, we have to analyze the relation between the affine Hecke algebra and the graded affine Hecke algebra as in [BM2], and also rely heavily on the classification of irreducible modules of the affine graded Hecke algebra in ([Lu1, Lu2, Lu3, Lu4]).

The main technical result that we prove in this direction is summarized in the following theorem. Let $\Psi = (X, X^\vee, R, R^\vee)$ be a root datum corresponding to a complex connected reductive group $G(\Psi)$. Let $\mathcal{H}^{\lambda, \lambda^*}(\Psi, z_0)$ be the affine Hecke algebra attached to Ψ (definition 2.1.1) and with parameters λ, λ^* of geometric type (section 3). We assume that z_0 is not a root of unity. Let $\mathcal{H}'(\Psi, z_0) = \mathcal{H}^{\lambda, \lambda^*}(\Psi, z_0) \rtimes \Gamma$ be an extended affine Hecke algebra by a finite group Γ acting by automorphisms of Ψ (definition 2.1.1).

Let $s_e \in T_e := X^\vee \otimes_{\mathbb{Z}} S^1$ be a fixed elliptic semisimple element in $G(\Psi)$.

Theorem 1.1.1. *The category of finite dimensional $\mathcal{H}'(\Psi, z_0)$ -modules whose central characters have elliptic parts $G(\Psi) \rtimes \Gamma$ -conjugate to s_e has the Vogan property (definition 4.3.1).*

This theorem plays the same role as its counterparts in [BM1] and [BM2]; namely it provides a means to conclude that whenever a finite dimensional representation of the Hecke algebra is unitary, then the corresponding infinite dimensional representation in the Bernstein component is also unitary.

We set up the machinery in this paper such that, in light of theorem 1.1.1, whenever certain, precise conditions on the Bernstein component are known to hold, the correspondence of Hermitian and unitary representations is automatic. The conditions that we need are the following:

- I. the Hecke algebra governing the Bernstein component is an extended affine Hecke algebra (definition 2.1.1) with parameters of geometric type (in the sense of Lusztig, see section 3), and
- II. the Hecke algebra isomorphism preserves the natural $*$ -structures, the tempered spectrum, and the Bernstein presentation (see section 5.4).

These conditions are known to hold in many cases, for example:

- (1) unramified principal series ([Bo1],[IM],[Ti]);
- (2) $GL(n, \mathbb{F})$ ([BK1]) and $GL(n, D)$ where D is a division algebra over \mathbb{F} ([Se]);
- (3) unipotent representations of simple groups of adjoint type ([Lu6, Lu7, Lu4],[BHK]);
- (4) ramified principal series of split groups ([Ro]);
- (5) pseudo-spherical principal series for double covers of split groups of simply-connected type ([LS]).

An important consequence of this approach is that it gives a unified framework for obtaining functorial unitary correspondences between two Bernstein components for possibly different groups. Whenever two Bernstein components are controlled by isomorphic affine Hecke algebras such that the conditions I, II above hold, one has a functorial unitary correspondence. The example we present in detail to illustrate this is the case of ramified principal series of a split group where we obtain a correspondence with the unramified principal series of a (split) endoscopic group (Theorem 1.4.1). But there is another, subtler phenomenon: there are important cases when the two affine Hecke algebras are not isomorphic, but certain of their graded versions (in the sense of [Lu1]) are. Again our methods allow us to conclude similar unitary correspondences. The example that we present in detail is that of unramified principal series of quasisplit unramified groups, and the correspondences are again with unramified principal series of certain (split) endoscopic groups (Theorem 1.3.1). We explain

the two examples in more detail in the introduction, after presenting the fundamental case of the Borel-Casselman equivalence.

1.2. Let \mathbb{F} be a p -adic field of characteristic zero. Let \mathcal{G} be the \mathbb{F} -points of a connected linear algebraic reductive group defined over \mathbb{F} . Let A be a maximally split torus of \mathcal{G} and set $M = C_{\mathcal{G}}(A)$, the centralizer of A in \mathcal{G} . Fix a special maximal compact open subgroup \mathcal{K} and an Iwahori subgroup $\mathcal{I} \subset \mathcal{K}$ of \mathcal{G} , attached by Bruhat-Tits theory to A . Let ${}^0M = M \cap \mathcal{K}$ be the maximal compact open subgroup of M . A complex character $\chi : M \rightarrow \mathbb{C}^\times$ is called unramified if $\chi|_{{}^0M} = 1$, and it is called ramified otherwise.

Fix ${}^0\chi$ a character of 0M , and consider the category $\mathfrak{R}^{0\chi}(\mathcal{G})$ of smooth admissible \mathcal{G} -representations, which appear as constituents of the minimal principal series induced from complex characters χ of M such that $\chi|_{{}^0M} = {}^0\chi$. This category is a Bernstein component of the category of all smooth admissible \mathcal{G} -representations. We are interested in the study of hermitian and unitary modules in $\mathfrak{R}^{0\chi}(\mathcal{G})$.

The first case is when ${}^0\chi = 1$, the unramified principal series. In this case, the category $\mathfrak{R}^1(\mathcal{G})$ is known to be naturally equivalent with the category of finite dimensional modules over the Iwahori-Hecke algebra $\mathcal{H}(\mathcal{G}, 1_{\mathcal{I}})$, (the Borel-Casselman correspondence, [Bo1]). The algebra $\mathcal{H}(\mathcal{G}, 1_{\mathcal{I}})$ is an affine Hecke algebra, possibly with unequal parameters (definition (2.1.1)), and has a natural $*$ -operation (section 2.6); therefore one can define hermitian and unitary modules.

Theorem 1.2.1. *In the Borel-Casselman equivalence $\mathfrak{R}^1(\mathcal{G}) \cong \mathcal{H}(\mathcal{G}, 1_{\mathcal{I}})$ -mod, the hermitian and unitary representations correspond, respectively.*

In this theorem, the group \mathcal{G} is of arbitrary isogeny and not necessarily split, and we emphasize that the correspondence is functorial. The case when \mathcal{G} is adjoint and split over \mathbb{F} is in [BM1, BM2]. In order to prove the claim, we need to extend the methods of Barbasch-Moy so that we cover extended affine (and affine graded in the sense of [Lu1]) Hecke algebras with unequal parameters.

1.3. Here is a first important consequence of our methods, indicative of endoscopy. Assume that \mathcal{G} is quasisplit quasisimple, and that it splits over an unramified extension of \mathbb{F} , and \mathcal{K} is hyperspecial. (When \mathcal{G} is simple of adjoint type, the Deligne-Langlands-Lusztig classification for the representations in $\mathfrak{R}^1(\mathcal{G})$ (and more generally, for unipotent representations) is in [Lu7].) We would like to relate the unitary representations in $\mathfrak{R}^1(\mathcal{G})$ with the unitary dual of certain split endoscopic groups. The Langlands complex dual group G is equipped with an automorphism τ of the root datum of G , defined by the inner class of \mathcal{G} (see section 6.1). (If \mathcal{G} is in fact split, then τ is trivial.) It is well-known that $W(\mathcal{G}, A)$ -conjugacy classes of unramified complex characters of M are in one-to-one correspondence with τ -twisted semisimple conjugacy classes in G . In this correspondence, if X is a subquotient of a principal series induced from an unramified character χ , we refer to the corresponding twisted semisimple conjugacy class in G as the infinitesimal character of X . We say that X has real infinitesimal character if the corresponding semisimple class is hyperbolic modulo the center.

Fix a semisimple elliptic element $s_e \in G$. Let $G(s_e\tau)$ denote the centralizer of s_e in G under τ -twisted conjugacy, a reductive group. When G is simply-connected (so \mathcal{G} is adjoint), $G(s_e\tau)$ is a connected group, but not in general. Let $\mathcal{G}(s_e\tau)$ denote the split \mathbb{F} -form of a (possibly disconnected) group dual to $G(s_e\tau)$. There is a natural one-to-one correspondence

between infinitesimal characters of \mathcal{G} with elliptic part τ -conjugate to s_e and real infinitesimal characters of $\mathcal{G}(s_e\tau)$. Moreover, we find that the affine graded at s_e Hecke algebras for \mathcal{G} and the affine graded at the identity Hecke algebra $\mathcal{G}(s_e\tau)$ are naturally isomorphic. The methods of this paper then imply the following result.

Theorem 1.3.1. *Let \mathcal{G} be a quasisplit quasisimple group which splits over an unramified extension of \mathbb{F} . Fix s_e an elliptic element in the dual complex group G . There is a natural one-to-one correspondence between irreducible representations in $\mathfrak{R}^1(\mathcal{G})$ whose infinitesimal character has elliptic part τ -conjugate with s_e , and representations in $\mathfrak{R}^1(\mathcal{G}(s_e\tau))$ with real infinitesimal character, such that the hermitian and unitary modules correspond, respectively.*

For example, if $s_e = 1$ and \mathcal{G} is the quasisplit form of the unitary group $PSU(2n)$ or $PSU(2n+1)$, then $\mathcal{G}(\tau)$ is the split form of $SO(2n+1)$ or $Sp(2n)$, respectively. In particular, we obtain a correspondence between the spherical unitary duals with real infinitesimal character of \mathcal{G} and $\mathcal{G}(\tau)$. This identification of spherical unitary duals (but by different methods) is also known to hold for the pairs of classical real groups $(U(n, n), SO(n+1, n))$ and $(U(n+1, n), Sp(2n, \mathbb{R}))$, see [Ba].

If \mathcal{G} does not split over an unramified extension of \mathbb{F} , one may apply the same method used in [Lu7], section 10.13, of identifying the Iwahori-Hecke algebra $\mathcal{H}(\mathcal{G}, 1_{\mathcal{I}'})$ and the category $\mathfrak{R}^1(\mathcal{G})$ with the Iwahori-Hecke algebra $\mathcal{H}(\mathcal{G}', 1_{\mathcal{I}'})$ and the category $\mathfrak{R}^1(\mathcal{G}')$, respectively, for a different group \mathcal{G}' which splits over an unramified extension of \mathbb{F} . (See [Lu7] for the list of pairs $(\mathcal{G}, \mathcal{G}')$.)

1.4. The second example is when ${}^0\chi$ is a nontrivial character of 0M . We rely on the theory of types results of [Ro] for ramified principal series, so we need to assume that \mathcal{G} is split, and have certain restrictions on the characteristic of \mathbb{F} . In this case too, our methods imply a correspondence of endoscopic type (see section 6.3). To ${}^0\chi$ one attaches a semisimple element $\widehat{{}^0\chi}$ in the Langlands dual G . Let $C_G(\widehat{{}^0\chi})$ be the centralizer of $\widehat{{}^0\chi}$ in G . This is a possibly disconnected reductive group. We define a dual split group $\mathcal{G}'({}^0\chi)$, the \mathbb{F} -points of a disconnected reductive group defined over \mathbb{F} (section 6.3).

Theorem 1.4.1. *Let \mathcal{G} be a split group and ${}^0\chi$ a nontrivial character of 0M . In the isomorphism of categories $\mathfrak{R}^{{}^0\chi}(\mathcal{G})$ and $\mathfrak{R}^1(\mathcal{G}'({}^0\chi))$ (from [Ro], see section 6.3), the hermitian and unitary representations correspond, respectively.*

Acknowledgements. This research was supported in part by the NSF: DMS 0554278 and 0901104 for D.B., DMS 0554278 for D.C.

2. AFFINE HECKE ALGEBRAS AND GRADED AFFINE HECKE ALGEBRAS

In this section we recall the definitions of the affine Hecke algebra, its graded version, and the relation between their unitary duals. We follow [Lu1] and [BM2]. There are certain minor modifications because we need to consider extended Hecke algebras.

2.1. Let $\Psi = (X, X^\vee, R, R^\vee)$ be a root datum. Thus X, X^\vee are two \mathbb{Z} -lattices with a perfect pairing $\langle \cdot, \cdot \rangle : X \times X^\vee \rightarrow \mathbb{Z}$, the subsets $R \subset X \setminus \{0\}$ and $R^\vee \subset X^\vee \setminus \{0\}$ are in bijection $\alpha \in R \longleftrightarrow \check{\alpha} \in R^\vee$, satisfying $\langle \alpha, \check{\alpha} \rangle = 2$. For every $\alpha \in R$, the reflections $s_\alpha : X \rightarrow X$, $s_\alpha(x) = x - \langle x, \check{\alpha} \rangle \alpha$, and $s_{\check{\alpha}} : X^\vee \rightarrow X^\vee$, $s_{\check{\alpha}}(y) = y - \langle \alpha, y \rangle \check{\alpha}$, leave R and R^\vee stable respectively. Let W be the finite Weyl group, *i.e.* the group generated by the set $\{s_\alpha : \alpha \in R\}$. It has a length function, which we denote by ℓ . We fix a choice of positive

roots R^+ , with basis Π of simple roots, and let $R^{\vee,+}$, Π^\vee be the corresponding images in R^\vee under $\alpha \mapsto \check{\alpha}$. We assume that Ψ is reduced (*i.e.*, $\alpha \in R$ implies $2\alpha \notin R$).

The connected complex linear reductive group corresponding to Ψ is denoted $G(\Psi)$ or just G if there is no danger of confusion. Then $T := X^\vee \otimes_{\mathbb{Z}} \mathbb{C}^\times$ is a maximal torus in G , and let $B \supset T$ be the Borel subgroup such that the roots of T in B are R^+ .

A parameter set for Ψ is a pair of functions (λ, λ^*) ,

$$\begin{aligned} \lambda &: \Pi \rightarrow \mathbb{Z}_{\geq 0}, \\ \lambda^* &: \{\alpha \in \Pi : \check{\alpha} \in 2X^\vee\} \rightarrow \mathbb{Z}_{\geq 0}, \end{aligned}$$

such that $\lambda(\alpha) = \lambda(\alpha')$ and $\lambda^*(\alpha) = \lambda^*(\alpha')$ whenever α, α' are W -conjugate.

Definition 2.1.1. The affine Hecke algebra $\mathcal{H}^{\lambda, \lambda^*}(\Psi, z)$, or just $\mathcal{H}(\Psi)$, associated to the root datum Ψ with parameter set (λ, λ^*) , is the associative algebra over $\mathbb{C}[z, z^{-1}]$ with unit (z is an indeterminate), defined by generators T_w , $w \in W$, and θ_x , $x \in X$ with relations:

$$(T_{s_\alpha} + 1)(T_{s_\alpha} - z^{2\lambda(\alpha)}) = 0, \text{ for all } \alpha \in \Pi, \quad (2.1.1)$$

$$T_w T_{w'} = T_{ww'}, \text{ for all } w, w' \in W, \quad (2.1.2)$$

$$\text{such that } \ell(ww') = \ell(w) + \ell(w'),$$

$$\theta_x \theta_{x'} = \theta_{x+x'}, \text{ for all } x, x' \in X, \quad (2.1.3)$$

$$\theta_x T_{s_\alpha} - T_{s_\alpha} \theta_{s_\alpha(x)} = (\theta_x - \theta_{s_\alpha(x)})(G(\alpha) - 1), \quad (2.1.4)$$

where $x \in X, \alpha \in \Pi$, and

$$\mathcal{G}(\alpha) = \begin{cases} \frac{\theta_\alpha z^{2\lambda(\alpha)} - 1}{\theta_\alpha - 1}, & \text{if } \check{\alpha} \notin 2X^\vee, \\ \frac{(\theta_\alpha z^{\lambda(\alpha) + \lambda^*(\alpha)} - 1)(\theta_\alpha z^{\lambda(\alpha) - \lambda^*(\alpha)} + 1)}{\theta_{2\alpha} - 1}, & \text{if } \check{\alpha} \in 2X^\vee. \end{cases} \quad (2.1.5)$$

Let Γ be a finite group endowed with a homomorphism $\Gamma \rightarrow \text{Aut}(G, B, T)$, satisfying the property that $\lambda(\gamma(\alpha)) = \lambda(\alpha)$ and $\lambda^*(\gamma(\alpha)) = \lambda^*(\alpha)$, for all $\gamma \in \Gamma$. Then we can form the extended affine Hecke algebra

$$\mathcal{H}'(\Psi) := \mathcal{H}(\Psi) \rtimes \Gamma, \quad (2.1.6)$$

by adding the generators $\{T_\gamma\}_{\gamma \in \Gamma}$ and relations

$$\begin{aligned} T_\gamma T_w &= T_{\gamma(w)} T_\gamma, \quad T_\gamma T_{\gamma'} = T_{\gamma\gamma'}, \quad T_\gamma \theta_x = \theta_{\gamma(x)} T_\gamma, \\ &\text{for } \gamma \in \Gamma, \quad w \in W, \quad x \in X. \end{aligned}$$

Remarks 2.1.2. (1) The case $\Pi^\vee \cap 2X^\vee \neq \emptyset$ can occur only if R has a factor of type B .

(2) Since $\theta_x - \theta_{s_\alpha(x)} = \theta_x(1 - \theta_{-\alpha}^n)$, where $n = \langle x, \check{\alpha} \rangle$, the denominator of $\mathcal{G}(\alpha)$ actually divides $\theta_x - \theta_{s_\alpha(x)}$.

(3) If $\lambda(\alpha) = c$ for all $\alpha \in \Pi$, and $\lambda^*(\alpha) = c$, for all $\check{\alpha} \in 2X^\vee$, we say that $\mathcal{H}(\Psi)$ is a Hecke algebra with equal parameters. For example, assume that $c = 1$, and z acts by \sqrt{q} . If Ψ corresponds to $SL(2, \mathbb{C})$, the algebra is generated by $T := T_{s_\alpha}$ and $\theta := \theta_{\frac{1}{2}\alpha}$, where α is the unique positive root, subject to

$$\begin{aligned} (T + 1)(T - q) &= 0; \\ \theta T - T\theta^{-1} &= (q - 1)\theta. \end{aligned} \quad (2.1.7)$$

On the other hand, if Ψ corresponds to $PGL(2, \mathbb{C})$, then the generators are $T := T_{s_\alpha}$ and $\theta' := \theta_\alpha$ subject to

$$\begin{aligned} (T+1)(T-q) &= 0; \\ \theta'T - T(\theta')^{-1} &= (q-1)(1+\theta'). \end{aligned} \tag{2.1.8}$$

2.2. Recall $T = X^\vee \otimes_{\mathbb{Z}} \mathbb{C}^\times$. Then

$$X = \text{Hom}(T, \mathbb{C}^\times), \quad X^\vee = \text{Hom}(\mathbb{C}^\times, T).$$

Let \mathcal{A} be the algebra of regular functions on $\mathbb{C}^\times \times T$. It can be identified with the abelian $\mathbb{C}[z, z^{-1}]$ -subalgebra of $\mathcal{H}(\Psi)$ generated by $\{\theta_x : x \in X\}$, where for every $x \in X$, $\theta_x : T \rightarrow \mathbb{C}^\times$ is defined by

$$\theta_x(y \otimes \zeta) = \zeta^{\langle x, y \rangle}, \quad y \in X^\vee, \quad \zeta \in \mathbb{C}^\times. \tag{2.2.1}$$

If we denote by \mathcal{H}_W the $\mathbb{C}[z, z^{-1}]$ -subalgebra generated by $\{T_w : w \in W\}$, then

$$\mathcal{H}(\Psi) = \mathcal{H}_W \otimes_{\mathbb{C}[z, z^{-1}]} \mathcal{A} \tag{2.2.2}$$

as a $\mathbb{C}[z, z^{-1}]$ -module. An important fact is that as algebras,

$$\mathcal{H}_W \cong \mathbb{C}[W] \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}].$$

In the case of an extended algebra, set $W' := W \rtimes \Gamma$ and $\mathcal{H}_{W'} := \mathcal{H}_W \rtimes \Gamma$. Then

$$\mathcal{H}'(\Psi) = \mathcal{H}_{W'} \otimes_{\mathbb{C}[z, z^{-1}]} \mathcal{A} \tag{2.2.3}$$

as a $\mathbb{C}[z, z^{-1}]$ -module.

Theorem 2.2.1 (Bernstein-Lusztig). *The center of $\mathcal{H}(\Psi)$ is $\mathcal{Z} = \mathcal{A}^W$, that is, the W -invariants in \mathcal{A} . Similarly, the center of $\mathcal{H}'(\Psi)$ is $\mathcal{A}^{W'}$.*

Let $\text{mod}\mathcal{H}(\Psi)$ or $\text{mod}\mathcal{H}'(\Psi)$ denote the categories of finite dimensional Hecke algebra modules. By Schur's lemma, every irreducible module (π, V) has a *central character*, i.e. there is a homomorphism $\chi : \mathcal{Z} \rightarrow \mathbb{C}$ such that $\pi(z)v = \chi(z)v$ for every $v \in V$ and $z \in \mathcal{Z}$. By theorem 2.2.1, the central characters correspond to W -conjugacy (respectively W' -conjugacy) classes $(z_0, s) \in \mathbb{C}^\times \times T$. Then, we have:

$$\text{mod}\mathcal{H}(\Psi) = \bigsqcup_{(z_0, s) \in \mathbb{C}^\times \times W \backslash T} \text{mod}_{(z_0, s)}\mathcal{H}(\Psi), \tag{2.2.4}$$

where $\text{mod}_{(z_0, s)}\mathcal{H}(\Psi)$ is the subcategory of modules with central character (corresponding to) (z_0, s) . Let $\text{Irr}_{(z_0, s)}\mathcal{H}(\Psi)$ be the set of isomorphism classes of simple objects in this category. One has the similar definitions for $\mathcal{H}'(\Psi)$. Throughout the section, we will assume that z_0 is a fixed number in $\mathbb{R}_{>1}$.

2.3. Fix a W -orbit \mathcal{O} of an element $\sigma \in T$, and denote by \mathcal{O}' the W' -orbit of σ . Then $\mathcal{O}' = (\mathcal{O}_1 = \mathcal{O}) \sqcup \mathcal{O}_2 \sqcup \dots \sqcup \mathcal{O}_m$ where each \mathcal{O}_i is a W -orbit. Define the decreasing chain of ideals \mathcal{I}^k , $k \geq 1$, in \mathcal{A} as

$$\mathcal{I} := \{f \text{ regular function on } \mathbb{C}^\times \times T : f(1, \sigma) = 0, \forall \sigma \in \mathcal{O}\}, \tag{2.3.1}$$

and \mathcal{I}^k the ideal of functions vanishing on \mathcal{O} to at least order k . Let $\widetilde{\mathcal{I}}^k := \mathcal{I}^k \mathcal{H} = \mathcal{H} \mathcal{I}^k$, $k \geq 1$, be the chain of ideals in $\mathcal{H}(\Psi)$, generated by the \mathcal{I}^k 's. Similarly define $(\mathcal{I}')^k$, and $\widetilde{(\mathcal{I}')^k}$ in $\mathcal{H}'(\Psi)$ using the orbit \mathcal{O}' .

Definition 2.3.1. The affine graded Hecke algebra $\mathbb{H}_{\mathcal{O}}(\Psi)$ is the associated graded object to the chain of ideals $\cdots \supset \widetilde{\mathcal{I}}^k \supset \cdots$ in $\mathcal{H}(\Psi)$. Similarly $\mathbb{H}_{\mathcal{O}'}(\Psi)$ is the graded object in $\mathcal{H}'(\Psi)$ for the filtration $\cdots \supset \widetilde{\mathcal{I}'}^k \supset \cdots$.

Let $\mathfrak{t} = X^{\vee} \otimes_{\mathbb{Z}} \mathbb{C}$ be the Lie algebra of T , and let $\mathfrak{t}^* = X \otimes_{\mathbb{Z}} \mathbb{C}$ be the dual space. Extend the pairing $\langle \cdot, \cdot \rangle$ to $\mathfrak{t}^* \times \mathfrak{t}$. Let \mathbb{A} be the algebra of regular functions on $\mathbb{C} \oplus \mathfrak{t}$. Note that \mathbb{A} can be identified with $\mathbb{C}[r] \otimes_{\mathbb{C}} S(\mathfrak{t}^*)$, where $S(\cdot)$ denotes the symmetric algebra, and r is an indeterminate. In the following δ denotes the delta function.

Theorem 2.3.2 ([Lu1, BM2]). *The graded Hecke algebra $\mathbb{H}_{\mathcal{O}}(\Psi)$ is a $\mathbb{C}[r]$ -algebra generated by $\{t_w : w \in W\}$, $S(\mathfrak{t}^*)$, and a set of orthogonal idempotents $\{E_{\sigma} : \sigma \in \mathcal{O}\}$ subject to the relations*

$$t_w \cdot t_{w'} = t_{ww'}, \quad w, w' \in W, \quad (2.3.2)$$

$$\sum_{\sigma \in \mathcal{O}} E_{\sigma} = 1, \quad E_{\sigma'} E_{\sigma''} = \delta_{\sigma', \sigma''} E_{\sigma'}, \quad E_{\sigma} t_{s_{\alpha}} = t_{s_{\alpha}} E_{s_{\alpha} \sigma}, \quad (2.3.3)$$

$$\omega \cdot t_{s_{\alpha}} - t_{s_{\alpha}} \cdot s_{\alpha}(\omega) = rg(\alpha) \langle \omega, \check{\alpha} \rangle, \quad \text{where } \alpha \in \Pi, \omega \in \mathfrak{t}^*, \quad (2.3.4)$$

$$g(\alpha) = \sum_{\sigma \in \mathcal{O}} E_{\sigma} \mu_{\sigma}(\alpha), \quad \text{and} \quad (2.3.5)$$

$$\mu_{\sigma}(\alpha) = \begin{cases} 0, & \text{if } s_{\alpha} \sigma \neq \sigma, \\ 2\lambda(\alpha), & \text{if } s_{\alpha} \sigma = \sigma, \quad \check{\alpha} \notin 2X^{\vee}, \\ \lambda(\alpha) + \lambda^*(\alpha) \theta_{-\alpha}(\sigma), & \text{if } s_{\alpha} \sigma = \sigma, \quad \check{\alpha} \in 2X^{\vee}. \end{cases} \quad (2.3.6)$$

Notice that if $s_{\alpha} \sigma = \sigma$, then $\theta_{\alpha}(\sigma) = \theta_{-\alpha}(\sigma)$, or equivalently, $\theta_{\alpha}(\sigma) \in \{\pm 1\}$. This implies that for the parameters $\mu_{\sigma}(\alpha)$ we have $\mu_{\sigma}(\alpha) \in \{0, 2\lambda(\alpha), \lambda(\alpha) - \lambda^*(\alpha)\}$, for every root $\alpha \in \Pi$. In particular, in the case of equal parameters Hecke algebra, the only possibilities are $\mu_{\sigma}(\alpha) \in \{0, 2\lambda(\alpha)\}$.

Remark 2.3.3. An important special case is when \mathcal{O} is formed of a single (W -invariant) element σ . Then there is only one idempotent generator $E_{\sigma} = 1$, so it is suppressed from the notation. The algebra $\mathbb{H}_{\mathcal{O}}(\Psi)$ is generated by $\{t_w : w \in W\}$ and $S(\mathfrak{t}^*)$ subject to the commutation relation

$$\omega \cdot t_{s_{\alpha}} - t_{s_{\alpha}} \cdot s_{\alpha}(\omega) = r \mu_{\sigma}(\alpha) \langle \omega, \check{\alpha} \rangle, \quad \alpha \in \Pi, \omega \in \mathfrak{t}^*, \quad (2.3.7)$$

where

$$\mu_{\sigma}(\alpha) = \begin{cases} 2\lambda(\alpha), & \text{if } \check{\alpha} \notin 2X^{\vee}, \\ \lambda(\alpha) + \lambda^*(\alpha) \theta_{-\alpha}(\sigma), & \text{if } \check{\alpha} \in 2X^{\vee}. \end{cases} \quad (2.3.8)$$

Still assuming that σ is W -invariant, we have $\theta_{\alpha}(\sigma) \in \{\pm 1\}$, for all $\alpha \in \Pi$. If in fact σ is in the center of group $G(\Psi)$, then $\theta_{\alpha}(\sigma) = 1$, for all $\alpha \in \Pi$.

Notation 2.3.4. We will use the notation $\mathbb{H}_{\mu_{\sigma}}$ for the graded algebra in the particular case defined by equations (2.3.7) and (2.3.8).

Example 2.3.5. Let Ψ be the root datum for $PGL(2, \mathbb{C})$, in the equal parameter case, and $\Gamma = \{1\}$. We present three cases:

(1) $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. This is clearly W -invariant. Then $\mathbb{H}_{\mu_{\sigma}}$ is generated by $t = t_{s_{\alpha}}$ and ω subject to

$$t^2 = 1, \quad t\omega + \omega t = 2r\lambda(\alpha). \quad (2.3.9)$$

(2) $\sigma = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. Since we are in $PGL(2, \mathbb{C})$, the element σ is W -invariant. Note that $\theta_\alpha(\sigma) = -1$, and so \mathbb{H}_{μ_σ} is generated by t and ω subject to

$$t^2 = 1, \quad t\omega + \omega t = 0. \quad (2.3.10)$$

(3) $\sigma = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$, $\zeta^2 \neq \pm 1$. Then $W \cdot \sigma = \{\sigma, \sigma^{-1}\}$. The algebra $\mathbb{H}_{\mathcal{O}}$ is generated by $E_\sigma, E_{\sigma^{-1}}, t, \omega$ satisfying the following relations:

$$\begin{aligned} E_\sigma^2 &= E_\sigma, \quad E_{\sigma^{-1}}^2 = E_{\sigma^{-1}}, \quad E_\sigma \cdot E_{\sigma^{-1}} = 0, \quad E_\sigma + E_{\sigma^{-1}} = 1, \\ tE_\sigma &= E_{\sigma^{-1}}t, \\ t^2 &= 1, \quad t\omega + \omega t = 0. \quad \square \end{aligned} \quad (2.3.11)$$

We can think of $\mathbb{H}_{\mathcal{O}}(\Psi)$ as the associative algebra defined by the relations (2.3.2)-(2.3.6). In particular if there is a homomorphism $\Gamma \rightarrow \text{Aut}(G, B, T)$, we can define

$$\mathbb{H}'_{\mathcal{O}}(\Psi) := \mathbb{H}_{\mathcal{O}}(\Psi) \rtimes \Gamma \quad (2.3.12)$$

as the associative algebra generated by $\{t_\gamma\}_{\gamma \in \Gamma}$ and the generators of $\mathbb{H}_{\mathcal{O}}$ satisfying (2.3.2)-(2.3.6), and in addition

$$\begin{aligned} t_\gamma t_w &= t_{\gamma(w)} t_\gamma, \quad t_\gamma t_{\gamma'} = t_{\gamma\gamma'}, \quad \gamma, \gamma' \in \Gamma, \\ t_\gamma \omega &= \gamma(\omega) t_\gamma, \quad w \in W, \omega \in \mathfrak{t}^*. \end{aligned} \quad (2.3.13)$$

Corollary 2.3.6. *There is a natural identification*

$$\mathbb{H}_{\mathcal{O}'}(\Psi) = \mathbb{H}'_{\mathcal{O}}(\Psi) (= \mathbb{H}_{\mathcal{O}}(\Psi) \rtimes \Gamma),$$

where $\mathbb{H}_{\mathcal{O}'}(\Psi), \mathbb{H}_{\mathcal{O}}(\Psi)$ are as in definition 2.3.1, and $\mathbb{H}'_{\mathcal{O}}(\Psi)$ is as in (2.3.12).

Proof. By (3) of proposition 3.2 of [BM2], $\mathbb{H}_{\mathcal{O}'} = \bigoplus_{i=1}^m \mathbb{H}_{\mathcal{O}_i}$; recall that $\{\mathcal{O}_i\}$ is the W -orbit partition of the W' -orbit \mathcal{O}' . Each T_γ induces (by grading) an algebra isomorphism

$$t_\gamma : \mathbb{H}_{\mathcal{O}_i} \rightarrow \mathbb{H}_{\gamma\mathcal{O}_i = \mathcal{O}_j} \quad (2.3.14)$$

and therefore an automorphism

$$t_\gamma : \mathbb{H}_{\mathcal{O}'} = \bigoplus \mathbb{H}_{\mathcal{O}_i} \rightarrow \bigoplus \mathbb{H}_{\gamma\mathcal{O}_i = \mathcal{O}_j} = \mathbb{H}_{\mathcal{O}'} \quad (2.3.15)$$

satisfying the required relations. We omit further details. \square

Define

$$C_{W'}(\sigma) := \{w \in W' : w\sigma = \sigma\}. \quad (2.3.16)$$

Then $W' \cdot \sigma = \{w_j \cdot \sigma : 1 \leq j \leq n\}$, where $\{w_1 = 1, w_2, \dots, w_n\}$ are coset representatives for $W'/C_{W'}(\sigma)$. Then $\{E_\tau\} = \{E_{w_j \cdot \sigma} : 1 \leq j \leq n\}$, and from theorem 2.3.1, $\mathbb{H}'_{\mathcal{O}} = \mathbb{C}[W'] \otimes (\mathcal{E}' \otimes \mathbb{A})$, as a $\mathbb{C}[r]$ -vector space, where \mathcal{E}' is the algebra generated by the $E_{w_j \cdot \sigma}$'s with $1 \leq j \leq n$.

Proposition 2.3.1 ([Lu1]). *The center of $\mathbb{H}_{\mathcal{O}}$ is $Z = (\mathcal{E} \otimes \mathbb{A})^W$. The center of $\mathbb{H}'_{\mathcal{O}}$ is $Z' = (\mathcal{E}' \otimes \mathbb{A})^{W'}$.*

It follows that the central characters of $\mathbb{H}'_{\mathcal{O}}$ are parameterized by $C_{W'}(\sigma)$ -orbits in $\mathbb{C} \oplus \mathfrak{t}$. Similarly to the last paragraph in section 2.2, define the category $\text{mod}_{(r_0, x)} \mathbb{H}'_{\mathcal{O}}$, where r acts by $r_0 > 0$ and $x \in \mathfrak{t}$.

2.4. We describe the structure of $\mathbb{H}_{\mathcal{O}'}(\Psi) = \mathbb{H}'_{\mathcal{O}}(\Psi)$ in more detail. Fix a $\sigma \in \mathcal{O}' \subset T$. We define a root datum $\Psi_\sigma = (X, R_\sigma, X^\vee, R_\sigma^\vee)$ with positive roots R_σ^+ , defined as follows:

$$R_\sigma = \left\{ \alpha \in R : \theta_\alpha(\sigma) = \begin{cases} 1, & \text{if } \check{\alpha} \notin 2X^\vee, \\ \pm 1, & \text{if } \check{\alpha} \in 2X^\vee \end{cases} \right\}, \quad (2.4.1)$$

$$R_\sigma^+ = R_\sigma \cap R^+, \quad (2.4.2)$$

$$R_\sigma^\vee = \{\check{\alpha} \in R^\vee : \alpha \in R_\sigma\}. \quad (2.4.3)$$

Note that Ψ_σ is the root datum for $(C_G(\sigma)_0, T)$, with positive roots R_σ^+ with respect to the Borel subgroup $C_G(\sigma)_0 \cap B$. Define

$$\Gamma_\sigma := \{w \in C_{W'}(\sigma) : w(R_\sigma^+) = R_\sigma^+\}. \quad (2.4.4)$$

This is the group of components $C_G(\sigma)/C_G(\sigma)_0$. There is a group homomorphism $\Gamma_\sigma \longrightarrow \text{Aut}(C_G(\sigma)_0, C_G(\sigma)_0 \cap B, T)$ such that $\mu_\sigma(\gamma\alpha) = \mu_\sigma(\alpha)$, so the extended Hecke algebra

$$\mathbb{H}'_{\mu_\sigma}(\Psi_\sigma) := \mathbb{H}_{\mu_\sigma}(\Psi_\sigma) \rtimes \Gamma_\sigma \quad (2.4.5)$$

is well defined.

Recall $\{w_1, \dots, w_n\}$ the coset representatives for $W'/C_{W'}(\sigma)$ from (2.3.16). Set

$$E_{i,j} = t_{w_i^{-1}w_j} E_{w_j\sigma} = E_{w_i\sigma} t_{w_i^{-1}w_j}, \text{ for all } 1 \leq i, j \leq n, \quad (2.4.6)$$

and let \mathcal{M}_n be the matrix algebra with basis $\{E_{i,j}\}$.

Theorem 2.4.1 ([Lu1]). *There is a natural algebra isomorphism*

$$\mathbb{H}'_{\mathcal{O}}(\Psi) \cong \mathcal{M}_n \otimes_{\mathbb{C}} \mathbb{H}'_{\mu_\sigma}(\Psi_\sigma) = \mathcal{M}_n \otimes_{\mathbb{C}} (\mathbb{H}_{\mu_\sigma}(\Psi_\sigma) \rtimes \Gamma_\sigma).$$

Since the only irreducible representation of \mathcal{M}_n is the n -dimensional standard representation, one obtains immediately the equivalences of categories:

$$\text{mod}_{(r_0, x)} \mathbb{H}'_{\mathcal{O}}(\Psi) \cong \text{mod}_{(r_0, x)} \mathbb{H}'_{\mu_\sigma}(\Psi_\sigma). \quad (2.4.7)$$

Remarks 2.4.2. (1) When $\Gamma = \{1\}$ and X^\vee is generated by R^\vee , that is, when Ψ is of simply connected type, or more generally, if X^\vee is generated by $R^\vee \cup \frac{1}{2}R^\vee$ (which includes the case of factors of type B as well), then $C_W(\sigma) \subset W_\sigma$, and so $\Gamma_\sigma = \{1\}$, for every $\sigma \in T$. In this case, $\mathbb{H}'_{\mathcal{O}}(\Psi_\sigma) = \mathbb{H}_{\mathcal{O}}(\Psi_\sigma)$, and there is no need to consider the extended graded Hecke algebras (2.4.5).

(2) When σ is W' -invariant, then $n = 1$, and so $\mathbb{H}'_{\mathcal{O}}(\Psi) \cong \mathbb{H}'_{\mu_\sigma}(\Psi_\sigma) = \mathbb{H}_{\mu_\sigma}(\Psi_\sigma) \rtimes \Gamma_\sigma$.

2.5. In this section, we discuss the relation between $\mathcal{H}(\Psi)$ and $\mathbb{H}_{\mathcal{O}}(\Psi)$. We need some definitions first.

The torus $T = X^\vee \otimes_{\mathbb{Z}} \mathbb{C}^\times$ admits a polar decomposition $T = T_e \times T_h$, where $T_e = X^\vee \otimes_{\mathbb{Z}} S^1$, and $T_h = X^\vee \otimes_{\mathbb{Z}} \mathbb{R}_{>0}$. Consequently, every $s \in T$ decomposes uniquely into $s = s_e \cdot s_h$, with $s_e \in T_e$ and $s_h \in T_h$. We call an element $s_e \in T_e$ elliptic, and an element $s_h \in T_h$ hyperbolic. Similarly, $\mathfrak{t} = X^\vee \otimes_{\mathbb{Z}} \mathbb{C}$ admits the decomposition $\mathfrak{t} = \mathfrak{t}_{i\mathbb{R}} \oplus \mathfrak{t}_{\mathbb{R}}$ into an imaginary part $\mathfrak{t}_{i\mathbb{R}} = X^\vee \otimes_{\mathbb{Z}} i\mathbb{R}$ and a real part $\mathfrak{t}_{\mathbb{R}} = X^\vee \otimes_{\mathbb{Z}} \mathbb{R}$.

We need to define certain completions of the Hecke algebras. The algebras $\mathbb{C}[r]$, \mathcal{S} , and $\mathbb{C}[r] \otimes \mathcal{S}$ consist of polynomial functions on \mathbb{C} , \mathfrak{t} and $\mathcal{M} := \mathbb{C} \oplus \mathfrak{t}$, respectively. Let $\widehat{\mathbb{C}}[r]$, $\widehat{\mathcal{S}}$,

and $\widehat{\mathbb{C}}[r] \otimes \widehat{\mathbb{S}}$ be the corresponding algebras of holomorphic functions. Let \mathcal{K} and $\widehat{\mathcal{K}}$ be the fields of rational and meromorphic functions on \mathcal{M} . Finally set $\widehat{\mathbb{A}} := \mathbb{A} \otimes_{\mathbb{S}} \widehat{\mathbb{S}} \subset \widehat{\mathcal{K}}$, and

$$\begin{aligned}\widehat{\mathbb{H}}_{\mathcal{O}} &:= \mathbb{C}[W] \otimes (\widehat{\mathbb{C}}[r] \otimes \widehat{\mathbb{A}}), \\ \widehat{\mathbb{H}}_{\mathcal{O}}(\mathcal{K}) &:= \mathbb{C}[W] \otimes (\mathcal{E} \otimes \mathcal{K}) \supset \widehat{\mathbb{H}}_{\mathcal{O}}, \\ \widehat{\mathbb{H}}_{\mathcal{O}}(\widehat{\mathcal{K}}) &:= \mathbb{C}[W] \otimes (\mathcal{E} \otimes \widehat{\mathcal{K}}) \supset \mathbb{H}_{\mathcal{O}}(\mathcal{K}), \widehat{\mathbb{H}}_{\mathcal{O}}.\end{aligned}$$

We make the analogous definitions for \mathbb{H}' .

Theorem 2.5.1 ([Lu1], 5.2 and [BM2] theorem 3.5). *The map*

$$\begin{aligned}\iota : \mathbb{C}[W] \rtimes (\mathcal{E} \otimes \widehat{\mathcal{K}}) &\longrightarrow \widehat{\mathbb{H}}_{\mathcal{O}}(\widehat{\mathcal{K}}), \text{ defined by} \\ \iota(E_{\sigma}) &= E_{\sigma}, \quad \iota(f) = f, \quad f \in \widehat{\mathcal{K}}, \\ \iota(t_{\alpha}) &= (t_{\alpha} + 1) \left(\sum_{\sigma \in \mathcal{O}} g_{\sigma}(\alpha)^{-1} E_{\sigma} \right) - 1,\end{aligned}$$

where

$$g_{\sigma}(\alpha) = 1 + \mu_{\sigma}(\alpha) \alpha^{-1} \in \widehat{\mathcal{K}}, \quad (2.5.1)$$

is an algebra isomorphism. Similarly for extended algebras, we have the analogous isomorphism $\iota' : \mathbb{C}[W'] \rtimes (\mathcal{E} \otimes \widehat{\mathcal{K}}) \longrightarrow \widehat{\mathbb{H}}'_{\mathcal{O}}(\widehat{\mathcal{K}})$, with $\iota'(\gamma) = \gamma$.

To every character χ of \mathcal{Z} (the center of $\mathcal{H} = \mathcal{H}_W \otimes \mathcal{A}$), there corresponds a maximal ideal $\mathcal{J}_{\chi} = \{z \in \mathcal{Z} : \chi(z) = 0\}$ of \mathcal{Z} . Define the quotients

$$\mathcal{A}_{\chi} = \mathcal{A} / \mathcal{A} \cdot \mathcal{J}_{\chi}, \quad \mathcal{H}(\Psi)_{\chi} = \mathcal{H}(\Psi) / \mathcal{H}(\Psi) \cdot \mathcal{J}_{\chi}. \quad (2.5.2)$$

Similarly, consider the ideal $\mathcal{I}_{\overline{\chi}}$ for every character $\overline{\chi}$ of Z in $\mathbb{H}_{\mathcal{O}}$, and define the analogous quotients. Then

$$\begin{aligned}\mathbb{A}_{\overline{\chi}} &:= \mathbb{A} / (\mathbb{A} \cdot \mathcal{I}_{\overline{\chi}}) = \widehat{\mathbb{A}} / (\widehat{\mathbb{A}} \cdot \widehat{\mathcal{I}}_{\overline{\chi}}) = \widehat{\mathbb{A}}_{\overline{\chi}}, \\ \mathbb{H}_{\overline{\chi}} &:= \mathbb{H}_{\mathcal{O}} / (\mathbb{H}_{\mathcal{O}} \cdot \mathcal{I}_{\overline{\chi}}) = \widehat{\mathbb{H}}_{\mathcal{O}} / (\widehat{\mathbb{H}}_{\mathcal{O}} \cdot \widehat{\mathcal{I}}_{\overline{\chi}}) = \widehat{\mathbb{H}}_{\overline{\chi}}.\end{aligned} \quad (2.5.3)$$

The similar definitions and formulas hold for \mathcal{H}' and $\mathbb{H}'_{\mathcal{O}}$.

The map

$$\tau : \mathbb{C} \oplus \mathfrak{t}_{\mathbb{R}} \rightarrow \mathbb{C}^{\times} \times T, \quad (r_0, \nu) \mapsto (z_0, s) = (e^{r_0}, \sigma \cdot e^{\nu}) \quad (2.5.4)$$

is $C_W(\sigma)$ -invariant. It matches the central characters

$$\tau : \overline{\chi} = C_W(\sigma) \cdot (r_0, \nu) \longleftrightarrow \chi = W \cdot (z_0, s). \quad (2.5.5)$$

Moreover, τ is a bijection onto the central characters of \mathcal{H} with elliptic part in \mathcal{O} . Similarly for extended algebras, we have a matching $\tau' : \overline{\chi}' = C_{W'}(\sigma) \cdot (r_0, \nu) \longleftrightarrow \chi = W' \cdot (z_0, s)$ which is a bijection onto central characters of \mathcal{H}' with elliptic part in \mathcal{O}' .

Proposition 2.5.1 ([BM2], proposition 4.1). *The map $\phi : \mathcal{A}[z, z^{-1}] \longrightarrow \widehat{\mathbb{C}}[r] \otimes \widehat{\mathbb{A}}$ defined by*

$$\begin{aligned}\phi(z) &= e^r, \\ \phi(\theta_x) &= \sum_{\sigma \in \mathcal{O}} \theta_x(\sigma) E_{\sigma} e^x, \quad x \in X,\end{aligned} \quad (2.5.6)$$

is a \mathbb{C} -algebra homomorphism which maps \mathcal{J}_{χ} to $\mathcal{I}_{\overline{\chi}}$ and defines by passage to the quotients an isomorphism between \mathcal{A}_{χ} and $\mathbb{A}_{\overline{\chi}}$.

The map $\Phi : \mathcal{H} \longrightarrow \widehat{\mathbb{H}}_{\mathcal{O}}(\widehat{\mathcal{K}})$ defined by

$$\begin{aligned}\Phi(a) &= \phi(a), \\ \Phi(T_\alpha + 1) &= \sum_{\sigma \in \mathcal{O}} E_\sigma(t_\alpha + 1) \phi(\mathcal{G}_\alpha) g_\sigma(\alpha)^{-1}\end{aligned}\tag{2.5.7}$$

with \mathcal{G}_α as in definition 2.1.1, and $g_\sigma(\alpha)$ as in (2.5.1), induces an isomorphism between \mathcal{H}_χ and $(\widehat{\mathbb{H}}_{\mathcal{O}})_{\overline{\chi}}$.

The map Φ depends on $(r_0, \nu) \in \mathcal{M}_{\mathbb{R}} (:= \mathbb{C} \oplus \mathfrak{t}_{\mathbb{R}})$. We write $\Phi_{(r_0, \nu)}$ when we want to emphasize this dependence.

Theorem 2.5.2 ([BM2], theorem 4.3). *Assume $\sigma \in T_e$. Let $\chi = W \cdot (e^{r_0}, \sigma \cdot e^\nu)$, $\overline{\chi} = C_W(\sigma) \cdot (r_0, \nu)$ be as in (2.5.5), with $(r_0, \nu) \in \mathbb{C} \oplus \mathfrak{t}_{\mathbb{R}}$. The isomorphism*

$$\Phi_{(r_0, \nu)} : \mathcal{H}(\Psi)_\chi \xrightarrow{\cong} \widehat{\mathbb{H}}_{\mathcal{O}}(\Psi)_{\overline{\chi}}$$

from (2.5.3) is analytic in $(r_0, \nu) \in \mathbb{C} \times \mathfrak{t}_{\mathbb{R}}$.

Both the proposition and the theorem hold with the obvious modifications for \mathcal{H}' and $\widehat{\mathbb{H}}'$.

2.6. The algebras $\mathcal{H}(\Psi)$ and $\widehat{\mathbb{H}}_{\mathcal{O}}(\Psi)$ have natural $*$ -operations. We recall their definitions on generators following [BM2].

For $\mathcal{H}(\Psi)$, the generators are $z, T_w, w \in W$, and $\theta_x, x \in X$ (definition 2.1.1), and we set:

$$z^* = z, \quad T_w^* = T_{w^{-1}}, \quad \theta_x^* = T_{w_0} \theta_{-x} T_{w_0}^{-1},\tag{2.6.1}$$

where w_0 is the longest Weyl group element. For \mathcal{H}' , $*$ acts in addition by

$$T_\gamma^* := T_{\gamma^{-1}}.\tag{2.6.2}$$

For $\widehat{\mathbb{H}}_{\mathcal{O}}(\Psi)$, recall that the generators are $r, t_w, w \in W, \omega \in t^*, E_{\sigma'}, \sigma' \in W \cdot \sigma$. The graded $*$ -operation is computed in [BM2]:

$$\begin{aligned}r^* &= r, \quad t_w^* = t_{w^{-1}}, \quad E_{\sigma'}^* = E_{\sigma'^*} \\ \omega^* &= -\omega + r \sum_{\beta \in R^+} t_{s_\beta} \langle \omega, \check{\beta} \rangle \sum_{\sigma' \in W \cdot \sigma} E_{\sigma'} \mu_{\sigma'}(\beta),\end{aligned}\tag{2.6.3}$$

where

$$\text{if } s = s_e \cdot s_h \in T = T_e \times T_h, \quad s^* := s_e \cdot s_h^{-1}.\tag{2.6.4}$$

Following [BM2], we call s such that $s^* \in W \cdot s$ hermitian. Note that every elliptic element is hermitian, and therefore, under our assumption that σ be elliptic, the $*$ -operation on $\widehat{\mathbb{H}}_{\mathcal{O}}(\Psi)$ is well-defined. For $\widehat{\mathbb{H}}'$, define in addition

$$t_\gamma^* = t_{\gamma^{-1}}.\tag{2.6.5}$$

Using these $*$ -operations, we define hermitian and unitary modules for $\mathcal{H}(\Psi)$ and $\widehat{\mathbb{H}}_{\mathcal{O}}(\Psi)$ as well as for the extended algebras $\mathcal{H}'(\Psi)$ and $\widehat{\mathbb{H}}'_{\mathcal{O}}(\Psi)$.

Proposition 2.6.1 ([BM2], 5.7). *In the setting of theorem 2.5.2, the isomorphism $\mathcal{H}(\Psi)_\chi \cong \widehat{\mathbb{H}}_{\mathcal{O}}(\Psi)_{\overline{\chi}}$ is compatible with the $*$ -structures. In particular, in the equivalence of categories*

$$\text{mod}_\chi \mathcal{H}(\Psi) \cong \text{mod}_{\overline{\chi}} \widehat{\mathbb{H}}_{\mathcal{O}}(\Psi),\tag{2.6.6}$$

$$\text{mod}_\chi \mathcal{H}'(\Psi) \cong \text{mod}_{\overline{\chi}} \widehat{\mathbb{H}}'_{\mathcal{O}}(\Psi),\tag{2.6.7}$$

the hermitian irreducible modules and the unitary irreducible modules correspond, respectively.

Corollary 2.6.1. *The equivalence of categories*

$$\mathcal{F} : \text{mod}_{\overline{\chi}} \mathbb{H}'_{\mu_\sigma}(\Psi_\sigma) \cong \text{mod}_{\chi} \mathcal{H}'(\Psi) \quad (2.6.8)$$

given by combining (2.6.6) and (2.4.7), takes hermitian irreducible modules to hermitian irreducible modules and unitary irreducible modules to unitary irreducible modules.

Proof. This is immediate from proposition 2.5.1, since tensoring with the n -dimensional standard representation of \mathcal{M}_n preserves irreducible, hermitian, and unitary modules, respectively. \square

Remark. Corollary 2.6.1 effectively says that, in order to compute the unitary dual for $\mathcal{H}'(\Psi)$, it is equivalent to compute the unitary dual with real central character for extended graded Hecke algebras $\mathbb{H}'_{\mu_\sigma}(\Psi_\sigma) = \mathbb{H}_{\mu_\sigma}(\Psi_\sigma) \rtimes \Gamma_\sigma$, for every conjugacy class of $\sigma \in T_e$ (elliptic semisimple elements). In the case when σ is a central element of G , this means that the unitary dual in $\text{mod}_{(z_0, s)} \mathcal{H}(\Psi)$ with $s_e = \sigma$ is identified with the unitary dual with real central character for $\mathbb{H}_1(\Psi_1)$. \square

For future purposes, we record here how the functor \mathcal{F} from (2.6.8) behaves with respect to the W' -structure. The first part is corollary 3.4.(2) in [BM2], and the second part is immediate from the remark after theorem 2.3.1.

Corollary 2.6.2 ([BM2],3.4). *As $\mathbb{C}[W']$ -modules:*

$$\mathcal{F}(\overline{V}) = \text{Ind}_{W'_\sigma}^{W'}(\overline{V}), \quad (2.6.9)$$

where $\overline{V} \in \text{mod}_{\overline{\chi}} \mathbb{H}'_{\mu_\sigma}(\Psi_\sigma)$, $\mathcal{F}(\overline{V}) \in \text{mod}_{\chi} \mathcal{H}(\Psi)$.

In particular, if σ is W' -invariant, then $\mathcal{F}(\overline{V}) \cong \overline{V}$ as $\mathbb{C}[W']$ -modules.

2.7. Let V be a $\mathcal{H}'(\Psi)$ -module on which z acts by $z_0 \in \mathbb{R}_{>1}$. For every $t \in T$, define a \mathcal{A} -generalized eigenspace of V :

$$V_t = \{v \in V : \text{for all } x \in X, (x(t) - \theta_x)^k v = 0, \text{ for some } k \geq 0\}. \quad (2.7.1)$$

We say that t is a weight of V if $V_t \neq \{0\}$. Let $\Sigma(V)$ denote the set of weights of V . We have $V = \bigoplus_{t \in \Sigma(V)} V_t$.

Definition 2.7.1. We say that V is tempered if, for all $x \in X^+ := \{x \in X : \langle x, \check{\alpha} \rangle \geq 0, \text{ for all } \alpha \in R^+\}$, and all $t \in \Sigma(V)$, $|x(t)| \leq 1$ holds.

Let \overline{V} be a $\mathbb{H}'_{\mu_\sigma}(\Psi_\sigma)$ -module on which r acts by $r_0 > 0$. For every $\nu \in \mathfrak{t}$, define a \mathbb{A} -generalized eigenspace of \overline{V} ,

$$\overline{V}_\nu = \{v \in \overline{V} : \forall \omega \in S(\mathfrak{t}^*), (\omega(\nu) - \omega)^k v = 0, k \gg 0\}. \quad (2.7.2)$$

We say that ν is a weight of \overline{V} if $\overline{V}_\nu \neq \{0\}$. Let $\Sigma(\overline{V})$ denote the set of weights of \overline{V} . We have

$$\overline{V} = \bigoplus_{\nu \in \Sigma(\overline{V})} \overline{V}_\nu.$$

Definition 2.7.2. We say that \overline{V} is tempered if, for all $\omega \in \mathbb{A}^+ := \{\omega \in S(\mathfrak{t}^*) : \langle \omega, \check{\alpha} \rangle \geq 0, \text{ for all } \alpha \in R^+\}$, and all $\nu \in \Sigma(\overline{V})$, we have $\langle \omega, \nu \rangle \leq 0$.

The two notions are naturally related.

Lemma 2.7.3 ([BM2, Lu4]). *In the equivalence of categories from corollary 2.6.2, the tempered modules correspond. Precisely, \overline{V} is tempered if and only if $V = \mathcal{F}(\overline{V})$ is tempered.*

3. GEOMETRIC GRADED HECKE ALGEBRAS AND LINEAR INDEPENDENCE

3.1. In this section we discuss the graded Hecke algebra $\mathbb{H} = \mathbb{H}_{\mu\sigma}(\Psi_\sigma)$ defined by the relations in (2.3.7) and (2.3.8) for the case $\sigma = 1$ (and $\Gamma = \{1\}$). We recall its definition in this particular case. Let (X, R, X^\vee, R^\vee) be a root datum for a reduced root system, with finite Weyl group W , and simple roots Π . The relations are in terms of the function μ in (2.3.8). The lattices X and X^\vee are not explicitly needed for the relations, just $\mathfrak{t} := X^\vee \otimes_{\mathbb{Z}} \mathbb{C}$ and $\mathfrak{t}^* = X \otimes_{\mathbb{Z}} \mathbb{C}$. Then $\mathbb{H} = \mathbb{H}^\mu(\mathfrak{t}^*, R)$ is generated over $\mathbb{C}[r]$ by $\{t_w : w \in W\}$ and $\omega \in \mathfrak{t}^*$, subject to

$$\begin{aligned} t_w \cdot t_{w'} &= t_{ww'}, & w, w' \in W, \\ \omega \cdot \omega' &= \omega' \cdot \omega, & \omega, \omega' \in \mathfrak{t}^*, \\ \omega \cdot t_{s_{\alpha_i}} - t_{s_{\alpha_i}} \cdot s_{\alpha}(\omega) &= 2r\mu_\alpha\langle \omega, \check{\alpha} \rangle, & \omega \in \mathfrak{t}^*, \alpha \in \Pi. \end{aligned} \tag{3.1.1}$$

We will consider only Hecke algebras of geometric type, that is those arising by the construction of [Lu1]. We will recall this next, but let us record first what the explicit cases are. Assume the root system is simple. Then there are at most two W -conjugacy classes in R^+ . Since μ is constant on W -conjugacy classes, it is determined by its values $\mu_s := \mu(\alpha_s)$ and $\mu_l := \mu(\alpha_l)$, where α_s is a short simple root, and α_l is a long simple root. For uniformity of notation in the table below, we say $\mu_s = \mu_l$ in the simply-laced case. Only the ratio of these parameters is important, and there are also obvious isomorphisms between types B and C . The list is in table 3.1.

Type	ratio μ_s/μ_l
A_n	1
B_n	$\mathbb{Z}_{>0}, \mathbb{Z}_{>0} + 1/2, \mathbb{Z}_{>0} + 1/4, \mathbb{Z}_{>0} + 3/4$
D_n	1
$E_{6,7,8}$	1
F_4	1, 2
G_2	1, 9

3.2. We review briefly the construction and classification of these algebras following the work of Lusztig. Let G be a complex connected reductive group, with a fixed Borel subgroup B , and maximal torus $H \subset B$. The Lie algebras will be denoted by the corresponding Gothic letters.

Definition 3.2.1. A cuspidal triple for G is a triple $(L, \mathcal{C}, \mathcal{L})$, where L is a Levi subgroup of G , \mathcal{C} is a nilpotent L -orbit on the Lie algebra \mathfrak{l} , and \mathcal{L} is an irreducible G -equivariant local system on \mathcal{C} which is cuspidal in the sense of [Lu5].

Let $\mathfrak{L}(G)$ denote the set of G -conjugacy classes of cuspidal triples for G . For example, $(H, 0, \text{triv}) \in \mathfrak{L}(G)$. Let us fix $(L, \mathcal{C}, \mathcal{L}) \in \mathfrak{L}(G)$, such that $H \subset L$, and $P = LU \supset B$ is a parabolic subgroup. Let T denote the identity component of the center of L . Set $W = N_G(L)/L$. This is a Coxeter group due to the particular form L must have to allow a cuspidal local system.

Definition 3.2.2. Let $\mathbb{H}(L, \mathcal{C}, \mathcal{L}) := \mathbb{H}^\mu(\mathfrak{t}^*, R)$ define a graded Hecke algebra as in (3.1.1) where:

- (i) \mathfrak{t} is the Lie algebra of T ;

- (ii) $W = N_G(L)/L$;
- (iii) R is the reduced part of the root system given by the nonzero weights of $\text{ad}(\mathfrak{t})$ on \mathfrak{g} ; it can be identified with the root system of the reductive part of $C_G(x)$, where $x \in \mathcal{C}$;
- (iv) R^+ is the subset of R for which the corresponding weight space lives in \mathfrak{u} ;
- (v) the simple roots $\Pi = \{\alpha_i : i \in I\}$ correspond to the Levi subgroups L_i containing L maximally: α_i is the unique element in R^+ which is trivial on the center of \mathfrak{l}_i ;
- (vi) for every simple α_i , $\mu_{\alpha_i} \geq 2$ is defined to be the smallest integer such that

$$\text{ad}(x)^{\mu_{\alpha_i}-1} : \mathfrak{l}_i \cap \mathfrak{u} \rightarrow \mathfrak{l}_i \cap \mathfrak{u} \text{ is zero.} \quad (3.2.1)$$

Up to constant scaling of the parameter function μ , all the algebras in table 3.1 appear in this way. The explicit classification of cuspidal triples when G is simple, along with the corresponding values for the parameters μ_α can be found in the tables of [Lu2], 2.13.

3.3. Consider the algebraic variety

$$\dot{\mathfrak{g}} = \{(x, gP) \in \mathfrak{g} \times G/P : \text{Ad}(g^{-1})x \in \mathcal{C} + \mathfrak{t} + \mathfrak{u}\}, \quad (3.3.1)$$

on which $G \times \mathbb{C}^\times$ acts via $(g_1, \lambda) : x \mapsto \lambda^{-2} \text{Ad}(g_1)x$, $x \in \mathfrak{g}$, and $gP \mapsto g_1 gP$, $g \in G$.

If \mathcal{V} is any $G \times \mathbb{C}^\times$ -stable subvariety of \mathfrak{g} , we denote by $\dot{\mathcal{V}}$ the preimage under the first projection $\text{pr}_1 : \dot{\mathfrak{g}} \rightarrow \mathfrak{g}$. Let \mathfrak{g}_N denote the variety of nilpotent elements in \mathfrak{g} . We will use the notation

$$\mathcal{P}_e := \{\dot{e}\} = \{gP : \text{Ad}(g^{-1})e \in \mathcal{C} + \mathfrak{u}\},$$

for any $e \in \mathfrak{g}_N$. (The identification is via the second projection $\dot{\mathfrak{g}} \rightarrow G/P$.) Define also $\mathcal{P}_e^s = \{gP \in \mathcal{P}_e : \text{Ad}(g^{-1})s \in \mathfrak{p}\}$.

We consider the projections $\text{pr}_{\mathcal{C}} : \dot{\mathfrak{g}} \rightarrow \mathcal{C}$, $\text{pr}_{\mathcal{C}}(x, g) = \text{pr}_{\mathcal{C}}(\text{Ad}(g^{-1})x)$, and $\text{pr}_P : \dot{\mathfrak{g}} \rightarrow \dot{\mathfrak{g}}$, $\text{pr}_P(x, g) = (x, gP)$, where $\tilde{\mathfrak{g}} = \{(x, g) \in \mathfrak{g} \times G : \text{Ad}(g^{-1})x \in \mathcal{C} + \mathfrak{h} + \mathfrak{u}\}$. They are both $G \times \mathbb{C}^\times$ -equivariant. Let $\dot{\mathcal{L}}$ be the $G \times \mathbb{C}^\times$ -equivariant local system on $\dot{\mathfrak{g}}$ defined by the condition $\text{pr}_{\mathcal{C}}^*(\dot{\mathcal{L}}) = \text{pr}_P^*(\dot{\mathcal{L}})$, and let $\dot{\mathcal{L}}^*$ be its dual local system.

The classification of simple modules for $\mathbb{H} = \mathbb{H}(L, \mathcal{C}, \dot{\mathcal{L}})$ is in [Lu2, Lu3, Lu4]. Let us fix a semisimple element $s \in \mathfrak{g}$ and $r_0 \in \mathbb{C}^\times$, and let $\mathcal{T} = \mathcal{T}_{s, r_0}$ be the smallest torus in $G \times \mathbb{C}^\times$ whose Lie algebra contains (s, r_0) . Let \mathfrak{g}_{2r_0} be the set of \mathcal{T} -fixed vectors in \mathfrak{g} , namely

$$\mathfrak{g}_{2r_0} = \{x \in \mathfrak{g} : [s, x] = 2r_0 x\}. \quad (3.3.2)$$

Let $C_G(s) \times \mathbb{C}^\times$ be the centralizer of (s, r_0) in $G \times \mathbb{C}^\times$. Since $s \in \mathfrak{g}$, $C_G(s)$ is a Levi subgroup of G , hence this centralizer is connected.

The construction of standard modules is in equivariant homology ([Lu2], section 1). For (e, ψ) , where $e \in \mathfrak{g}_{2r_0}$, and $\psi \in \widehat{A}(s, e)$, the standard geometric module is (see [Lu3], 10.7, 10.12) :

$$\begin{aligned} X_{(s, e, \psi)} &= \text{Hom}_{A(s, e)}[\psi : H_\bullet^{\{1\}}(\mathcal{P}_e^s, \dot{\mathcal{L}})] \\ &= \text{Hom}_{A(s, e)}[\psi : \mathbb{C}_{(s, r_0)} \otimes_{H_\bullet^{C_G(e)^0}} H_\bullet^{C_G \times \mathbb{C}^*(e)}(\mathcal{P}_e, \dot{\mathcal{L}})]. \end{aligned} \quad (3.3.3)$$

One considers the action of $A_G(e) = C_G(e)/C_G(e)^0$ on the homology $H_\bullet^{\{1\}}(\mathcal{P}_e, \dot{\mathcal{L}})$, and let $\widehat{A}(e)_\mathcal{L}$ denote the representations of $A(e)$ which appear in this way. Note that the natural map $A(s, e) \rightarrow A(e)$ is in fact an injection. Let $\widehat{A}(s, e)_\mathcal{L}$ denote the representations of $A(s, e)$

which appear as restrictions of $\widehat{A}(e)_{\mathcal{L}}$ to $A(s, e)$. By [Lu3], 8.17, and [Lu2], 8.10, $X_{(s,e,\psi)} \neq 0$ if and only if $\psi \in \widehat{A}(s, e)_{\mathcal{L}}$. One can phrase the classification as follows.

Theorem 3.3.1 ([Lu3] 8.10, 8.14, 8.17, [Lu4] 1.15). *A standard module $X_{(s,e,\psi)}$ is nonzero if and only if $\psi \in \widehat{A}(s, e)_{\mathcal{L}}$. When $X_{(s,e,\psi)} \neq 0$, it has a unique irreducible quotient $L_{(s,e,\psi)}$. This induces a natural one-to-one correspondence*

$$\begin{aligned} \text{Irr}_{r_0} \mathbb{H}(L, \mathcal{C}, \mathcal{L}) &\leftrightarrow \{(s, e, \psi) : [s, e] = r_0 e, \\ &s \in \mathfrak{g} \text{ semisimple}, e \in \mathfrak{g}_N, \psi \in \widehat{A}(s, e)_{\mathcal{L}}\} / G. \end{aligned} \quad (3.3.4)$$

3.4. We are interested in the W -structure of standard modules. Let $e \in \mathfrak{g}_N$ be given. By [Lu5] (also [Lu8] section 24), the homology group $H_{\bullet}^{\{1\}}(\mathcal{P}_e, \dot{\mathcal{L}})$ carries a $A(e) \times W$ action. Moreover,

$$\mu(e, \phi) := \text{Hom}_{A(e)}[\phi, H_0^{\{1\}}(\mathcal{P}_e, \dot{\mathcal{L}})] \quad (3.4.1)$$

is an irreducible W -representation. The correspondence $\widehat{A}(e)_{\mathcal{L}} \rightarrow \widehat{W}$, $(e, \phi) \rightarrow \mu(e, \phi)$ is the generalized Springer correspondence of [Lu5], and it is a bijection. We summarize the relevant results from [Lu5, Lu8] that we will need in the following statement.

Theorem 3.4.1 ([Lu5], 6.5, [Lu8], 24.4). *Let $(L, \mathcal{C}, \mathcal{L})$ be a cuspidal triple as before.*

- (1) *If $e \in \mathfrak{g}_N$, $\phi \in \widehat{A}(e)_{\mathcal{L}}$, then $\mu(e, \phi)$ from (3.4.1) is an irreducible W -representation.*
- (2) *The W -representation $\mu(e, \phi)$ appears with multiplicity one in $H_{\bullet}^{\{1\}}(\mathcal{P}_e, \dot{\mathcal{L}})$.*
- (3) *If the W -representation $\mu(e', \phi')$ occurs in $H_{\bullet}^{\{1\}}(\mathcal{P}_e, \dot{\mathcal{L}})$, then $e' \in \overline{G \cdot e}$, and if $G \cdot e' = G \cdot e$, then necessarily $\phi' = \phi$.*
- (4) *If $\phi' \notin \widehat{A}(e)_{\mathcal{L}}$, then $\text{Hom}_{A(e)}[\phi', H_{\bullet}^{\{1\}}(\mathcal{P}_e, \dot{\mathcal{L}})] = 0$.*

Claim (4) in this theorem is proved in [Lu8], as a consequence of a (generalized) Green polynomials algorithm.

To transfer these results to the W -structure of standard modules, via (3.3.3), we also need the deformation argument of [Lu3], 10.13.

Lemma 3.4.2 ([Lu3], 10.13). *In the notation of (3.3.3), there is an isomorphism of W -representations*

$$\begin{aligned} X_{(s,e,\psi)} &= \text{Hom}_{A(s,e)}[\psi : H_{\bullet}^{\{1\}}(\mathcal{P}_e^s, \dot{\mathcal{L}})] = \\ &= \text{Hom}_{A(s,e)}[\psi : H_{\bullet}^{\{1\}}(\mathcal{P}_e, \dot{\mathcal{L}})]. \end{aligned} \quad (3.4.2)$$

3.5. Now we can recall the classification of tempered modules for the geometric Hecke algebras as in [Lu4]. Fix $r_0 > 0$.

Definition 3.5.1. A semisimple element $\sigma \in \mathfrak{g}$ is called hyperbolic (resp. elliptic) if $\text{ad}(\sigma) : \mathfrak{g} \rightarrow \mathfrak{g}$ has only real (resp. imaginary) eigenvalues.

Theorem 3.5.2 ([Lu4], 1.21). *The module $L_{(s,e,\psi)}$, where $\psi \in \widehat{A}(s, e)_{\mathcal{L}}$, is a tempered module for $\mathbb{H}(L, \mathcal{C}, \mathcal{L})$ if and only if there exists a Lie triple $\{e, h, f\}$ in \mathfrak{g} , such that $[s, h] = 0$, $[s, f] = -2r_0 f$, $[s, e] = 2r_0 e$, and $s - r_0 h$ is elliptic. In this case, $L_{(s,e,\psi)} = X_{(s,e,\psi)}$.*

By $sl(2)$ -theory, any middle element h of a Lie triple is hyperbolic. The condition that $s - r_0 h$ be elliptic, implies that if s is hyperbolic, then in fact $s = r_0 h$. In this case $A(e) = A(s, e)$. Recall also that there is a one-to-one correspondence between nilpotent G -orbits in \mathfrak{g} and G -conjugacy classes of Lie triples. Finally, we may assume that s is in \mathfrak{t} , and therefore

the central character of the irreducible $\mathbb{H}(L, \mathcal{C}, \mathcal{L})$ -module $L_{(s, e, \psi)}$ is the (W -conjugacy class of the) projection on \mathfrak{t} of s . (The notation is as in section 3.2.) Putting these together, we have the following corollary.

Corollary 3.5.3. *The map*

$$\{(e, \phi) : e \in \mathfrak{g}_N, \phi \in \widehat{A}(e)_{\mathcal{L}}\} / G \longleftrightarrow X_{(r_0 h, e, \phi)}, \quad (3.5.1)$$

is a one-to-one correspondence onto the set of isomorphism classes of tempered irreducible $\mathbb{H}(L, \mathcal{C}, \mathcal{L})$ -modules with real central character, on which r acts by r_0 .

We can now formulate the main result we will need in the rest of the paper. This is the generalization of the result in [BM1] for Hecke algebras with equal parameters, and it also appeared in an equivalent form in [Ci].

Proposition 3.5.1. *Assume $r_0 \neq 0$. The lowest W -type correspondence*

$$X_{(r_0 h, e, \phi)} \longrightarrow \mu(e, \phi) \quad (3.5.2)$$

between tempered modules in $\text{Irr}_{r_0} \mathbb{H}(L, \mathcal{C}, \mathcal{L})$ with real central character and \widehat{W} is a bijection. Moreover, the map

$$X_{(r_0 h, e, \phi)} \longrightarrow X_{(r_0 h, e, \phi)}|_W \quad (3.5.3)$$

is uni-triangular with respect to the closure ordering of nilpotent orbits and the lowest W -type map. In particular, the set of tempered modules with real central character in $\text{Irr}_{r_0} \mathbb{H}(L, \mathcal{C}, \mathcal{L})$ are linearly independent in the Grothendieck group of W .

Proof. Using lemma 3.4.2 and theorem 3.4.1, we see that every tempered module $X_{(r_0 h, e, \phi)}$ as in corollary 3.5.3 has a unique lowest W -type (with respect to the closure ordering of nilpotent orbits) $\mu(e, \phi)$, and this appears with multiplicity one. The unitriangularity is also clear from the same results. \square

3.6. We would like to extend proposition 3.5.1 to affine graded Hecke algebras $\mathbb{H}' := \mathbb{H} \rtimes \Gamma$ of the type as in section 2.4, under the assumption that \mathbb{H} is of geometric type.

We wish to study the W' -structure of tempered \mathbb{H}' -modules with real central character. For this, we need some elements of Clifford-Mackey theory, as in [RR]. We recall the general setting, for an extended algebra $K' := K \rtimes \Gamma$, which will then be specialized first to $K = \mathbb{C}[W]$ and $K' = \mathbb{C}[W']$, and then to $K = \mathbb{H}$ and $K' = \mathbb{H}'$.

Let K be a finite dimensional \mathbb{C} -algebra, with an action by algebra automorphisms by Γ , and set $K' = K \rtimes \Gamma$. If V is a finite dimensional module of K , and $\gamma \in \Gamma$, then let γV denote the K -module with the action $x \circ v := \gamma^{-1}(x)v$, $x \in K$, $v \in V$. Clearly, V is irreducible if and only if γV is irreducible. Assume V is an irreducible K -module. Define the inertia subgroup

$$\Gamma_V = \{\gamma \in \Gamma : V \cong^{\gamma} V\}. \quad (3.6.1)$$

Since V is simple, the isomorphism $V \rightarrow^{\gamma} V$, for $\gamma \in \Gamma_V$, is unique up to a scalar multiple. We fix a family of isomorphisms $\{\tau_{\gamma} : V \rightarrow^{\gamma^{-1}} V\}_{\gamma \in \Gamma_V}$, and define the factor set

$$\beta : \Gamma_V \times \Gamma_V \rightarrow \mathbb{C}^*, \quad \text{such that } \tau_{\gamma} \tau_{\gamma'} = \beta(\gamma, \gamma') \tau_{\gamma \gamma'}. \quad (3.6.2)$$

Let $(\mathbb{C}\Gamma_V)_{\beta^{-1}}$ be the algebra with basis $\{\gamma : \gamma \in \Gamma_V\}$ and multiplication

$$\gamma \cdot \gamma' = \beta(\gamma, \gamma')^{-1} (\gamma \gamma'), \quad (3.6.3)$$

where the latter multiplication is understood in $\mathbb{C}\Gamma_V$. Up to algebra isomorphism, the algebra $(\mathbb{C}\Gamma_V)_{\beta^{-1}}$ is independent of the choice of the family $\{\tau_{\gamma}\}$. If U is any irreducible

$(\mathbb{C}\Gamma_V)_{\beta^{-1}}$ -module, then there is a natural $K \rtimes \Gamma_V$ action on $V \otimes U$: $x\gamma(v \otimes u) := (x\tau_\gamma v) \otimes (\gamma u)$, for $x \in K$, $\gamma \in \Gamma_V$.

Theorem 3.6.1 ([RR],A.6). *With the same notation as above, define the induced module*

$$V \rtimes U := \text{Ind}_{K \rtimes \Gamma_V}^{K \rtimes \Gamma} (V \otimes U).$$

Then

- (a) $V \rtimes U$ is an irreducible $K \rtimes \Gamma$ -module.
- (b) Every irreducible $K \rtimes \Gamma$ -module appears in this way.
- (c) If $V \rtimes U \cong V' \rtimes U'$, then V, V' are Γ -conjugate, and $U \cong U'$ as $(\mathbb{C}\Gamma_V)_{\beta^{-1}}$ -modules.

We go back to our setting. Set $K = \mathbb{C}[W]$ first. For every $\mu \in \widehat{W}$, let Γ_μ be the inertia group, and fix a family of isomorphisms $\{a_\gamma^\mu : \mu \rightarrow \gamma^{-1}\mu\}_{\gamma \in \Gamma_\mu}$. Since the action of Γ on W comes from the action of Γ on the root datum, we have the following lemma.

Lemma 3.6.2. *In the notation of theorem 3.4.1, if $\mu = \mu(e, \phi) \in \widehat{W}$, for some $e \in \mathfrak{g}_N$ and $\phi \in \widehat{A}(e)_\mathcal{L}$, then for every $\gamma \in \Gamma$ we have $\gamma\mu = \mu(\gamma e, \gamma\phi)$.*

When \mathbb{H} has equal parameters, this statement is a particular case of propositions 2.6.1 and 2.7.3 in [Re]. In more generality, one can follow the analogous argument using the construction of the W -action from [Lu2] 3.4, 8.1, and the formal functorial properties of equivariant homology from section 1 in [Lu2].

Now, let us specialize to $K = \mathbb{H}$, so that $K' = \mathbb{H}'$. Assume V is a tempered module of \mathbb{H} with real central character. Then also \mathcal{V} is tempered with real central character for any $\gamma \in \Gamma$. By proposition 3.5.1, there exists a unique lowest W -type of V , call it $\mu = \mu(e, \phi)$, which appears with multiplicity one. To emphasize this correspondence, we write $V = V(e, \phi)$.

Proposition 3.6.1. *We have $\Gamma_{V(e, \phi)} = \Gamma_{\mu(e, \phi)}$. Moreover, the factor set for $\Gamma_{V(e, \phi)}$ can be chosen to be equal to the factor set for $\Gamma_{\mu(e, \phi)}$.*

Proof. Assume the W -structure of $V(e, \phi)$ is

$$V(e, \phi)|_W = \mu(e, \phi) \oplus \bigoplus_{e \in \overline{G \cdot e'}} m_{(e', \phi')} \mu(e', \phi'). \quad (3.6.4)$$

Let $\gamma \in \Gamma$ be given. Then

$$\begin{aligned} \gamma V|_W &= \mu(\gamma e, \gamma\phi) \oplus \bigoplus_{e \in \overline{G \cdot e'}} m_{(e', \phi')} \mu(\gamma e', \gamma\phi') \\ &= \mu(\gamma e, \gamma\phi) \oplus \bigoplus_{\gamma e \in \overline{G \cdot \gamma e'}} m_{(e', \phi')} \mu(\gamma e', \gamma\phi'), \end{aligned} \quad (3.6.5)$$

where one uses the obvious fact that $e \in \overline{G \cdot e'}$ if and only if $\gamma e \in \overline{G \cdot \gamma e'}$. This means that the lowest W -type of γV is $\gamma\mu$. By proposition 3.5.1, it follows that $V \cong^\gamma V$ if and only if $\mu \cong^\gamma \mu$. This proves the first claim in the lemma.

For the second claim, let β be the factor set for μ corresponding to the isomorphisms $\{a_\gamma^\mu\}_{\gamma \in \Gamma_\mu}$. Let $\{\tau_\gamma : V \rightarrow \gamma^{-1}V\}_{\gamma \in \Gamma_V}$ be a family of isomorphisms for V . Then by restriction to Hom-spaces, we get

$$\text{Hom}_W[\mu : V] \xrightarrow{\tau_\gamma} \text{Hom}_W[\mu : \gamma^{-1}V] \xrightarrow{a_\gamma} \text{Hom}_W[\gamma^{-1}\mu : \gamma^{-1}V]. \quad (3.6.6)$$

By theorem 3.4.1(2), these spaces are one-dimensional, and so the composition is a scalar. We normalize τ_γ so that this scalar equals to one. This forces $\{\tau_\gamma\}$ to have the same factor set β as $\{a_\gamma\}$. \square

Corollary 3.6.3. ¹ *There is a one-to-one correspondence $V' = V(e, \phi) \rtimes U \rightarrow \mu'_{V'} = \mu(e, \phi) \rtimes U$, $U \in \widehat{\Gamma}_{\mu(e, \phi)}$, between tempered modules with real central character for $\mathbb{H}' = \mathbb{H} \rtimes \Gamma$ and representations of $W' = W \rtimes \Gamma$. Moreover, in the Grothendieck group of W' , the set of tempered \mathbb{H}' -modules with real central character is linearly independent.*

Proof. The first claim follows immediately from proposition 3.6.1, and also the fact that $\mu'_{V'}$ appears with multiplicity one in V' .

The second claim follows immediately, once we define a partial ordering on \widehat{W}' by setting $\mu'_{(e_1, \phi_1)} < \mu'_{(e_2, \phi_2)}$ if and only if $e_1 \in \overline{G} \cdot e_2$. Then $\mu'_{V'}$ is the lowest W' -type of V' with respect to this order, and the restriction map $V' \rightarrow V'|_{W'}$ is uni-triangular. \square

4. JANTZEN FILTRATIONS AND SIGNATURE CHARACTERS

In the previous sections, we encountered three types of “graded” Hecke algebras associated to a root system, which appear naturally in the reductions of section 2.3 from the affine Hecke algebra attached to a root datum. There is the usual graded Hecke algebra \mathbb{H} , definition (3.1.1), the extended Hecke algebra $\mathbb{H}' = \mathbb{H} \rtimes \Gamma$, by a group Γ of automorphisms of the root system for \mathbb{H} , and also, an induced Hecke algebra, which we will denote now $\widetilde{\mathbb{H}}' = \mathcal{M}_n \otimes_{\mathbb{C}} \mathbb{H}'$, where $\mathcal{M}_n = \{E_{i,j}\}$ is a matrix algebra. The finite group parts of these algebras will be denoted by W , $W' = W \rtimes \Gamma$, and $\widetilde{W}' = \langle W', E_{i,i} : i = 1, n \rangle$, respectively.

4.1. The Langlands classification for \mathbb{H} is in [Ev], theorem 2.1. We formulate it in the setting of $\mathbb{H}' = \mathbb{H} \rtimes \Gamma$. The proof is the same.

Recall that the graded Hecke algebra \mathbb{H} corresponds to a root system $(\mathfrak{t}^*, R, \mathfrak{t}, R^\vee)$, simple roots Π , and parameter set μ . Assume the indeterminate r acts by $r_0 \neq 0$. Let $\Pi_P \subset \Pi$ be given, then we define a parabolic subalgebra $\mathbb{H}_P \subset \mathbb{H}$, which, as a \mathbb{C} -vector space is $\mathbb{H}_P = \mathbb{C}[W_P] \otimes S(\mathfrak{t}^*)$, where $W_P \subset W$ is the subgroup generated by the reflection in the roots of Π_P . The parameter set on \mathbb{H}_P is obtained by restriction from μ . Define

$$\begin{aligned} \mathfrak{a}_P &= \{x \in \mathfrak{t} : \langle \alpha, x \rangle = 0, \forall \alpha \in \Pi_P\}, \\ \mathfrak{a}_P^* &= \{\omega \in \mathfrak{t}^* : \langle \omega, \check{\alpha} \rangle = 0, \forall \alpha \in \Pi_P\}, \\ \mathfrak{t}_M &= \{\omega \in \mathfrak{t}^* : \langle \omega, x \rangle = 0, \forall x \in \mathfrak{a}_P^*\}, \\ \mathfrak{t}_M^* &= \{x \in \mathfrak{t} : \langle \omega, x \rangle = 0, \forall \omega \in \mathfrak{a}_P^*\}. \end{aligned}$$

Then $\mathbb{H}_P = \mathbb{H}_M \otimes S(\mathfrak{a}_P^*)$, where $\mathbb{H}_M = \mathbb{C}[W_P] \otimes S(\mathfrak{t}_M^*)$ as a vector space (and defining commutation relations coming from \mathbb{H}).

To define the corresponding parabolic subalgebra of \mathbb{H}' , set

$$\Gamma_P = \{\gamma \in \Gamma : \gamma \cdot \Pi_P = \Pi_P\}, \quad (4.1.1)$$

and then define $\mathbb{H}'_P = \mathbb{H}_P \rtimes \Gamma_P$ and $\mathbb{H}'_M = \mathbb{H}_M \rtimes \Gamma_P$. Then we have $\mathbb{H}'_P = \mathbb{H}'_M \otimes S(\mathfrak{a}_P^*)$.

¹Recently, the preprint [So] announced a proof, by different methods, of the linear W' -independence of tempered modules with real central character, but not the “lowest W' -type” uni-triangularity, for affine graded Hecke algebras with arbitrary parameters.

Theorem 4.1.1 ([Ev], 2.1).

- (i) Let V' be an irreducible \mathbb{H}' -module. Then V' is a quotient of a standard induced module $I(P, U', \nu) := \mathbb{H}' \otimes_{\mathbb{H}'_P} (U' \otimes \mathbb{C}_\nu)$, where U' is a tempered \mathbb{H}'_M -module, and $\nu \in \mathfrak{a}_P$ is such that $\langle \alpha, \operatorname{Re} \nu \rangle > 0$, for all $\alpha \in \Pi \setminus \Pi_P$.
- (ii) Any standard module $I(P, U', \nu)$ as in (i) has a unique irreducible quotient, denoted $J(P, U', \nu)$.
- (iii) $J(P_1, U'_1, \nu_1) \cong J(P_2, U'_2, \nu_2)$ if and only if $\Pi_{P_1} = \Pi_{P_2}$, $U'_1 \cong U'_2$ as \mathbb{H}_M -modules, and $\nu_1 = \nu_2$.

One can transfer the classification to $\widetilde{\mathbb{H}'}$ as well, via the functor

$$\mathcal{T} : (\mathbb{H}' - \text{mod}) \rightarrow (\widetilde{\mathbb{H}'} - \text{mod}), \quad V' \mapsto \widetilde{V}' := \mathbb{C}^n \otimes_{\mathbb{C}} V', \quad (4.1.2)$$

where $\mathbb{C}^n = \operatorname{span}_{\mathbb{C}}\{E_{i,1} : i = 1, n\}$ is, up to isomorphism, the unique irreducible module of \mathcal{M}_n .

4.2. Let \mathcal{K} denote any one of the three algebras and let \mathcal{W} denote its finite group part. For every irreducible \mathcal{K} -module, we define the notions of \mathcal{W} -character and signature. The idea is due to [Vo], and it was used in the p -adic and Hecke algebra cases by [BM1].

Definition 4.2.1 (1). Let (π, V) be a finite dimensional \mathcal{K} -module. For every irreducible representation (δ, V_δ) of \mathcal{W} , set $V(\delta) = \operatorname{Hom}_{\mathcal{W}}[V_\delta, V]$, and let $m(\delta) = \dim_{\mathbb{C}} V(\delta)$ be the multiplicity of δ in V . We define the \mathcal{W} -character of (π, V) to be the formal combination

$$\theta_{\mathcal{W}}(V) = \sum_{\delta \in \widehat{\mathcal{W}}} m(\delta) \delta. \quad (4.2.1)$$

Recall from section 2 that the algebra \mathcal{K} has a $*$ -operation (defined explicitly), and that we defined hermitian and unitary \mathcal{K} -modules with respect to it.

Definition 4.2.2 (2). Let (π, V) be a finite dimensional \mathcal{K} -module with a nondegenerate hermitian form $\langle \cdot, \cdot \rangle$. For every $(\delta, V_\delta) \in \widehat{\mathcal{W}}$, fix a positive definite hermitian form on V_δ . Then the space $V(\delta)$ acquires a nondegenerate hermitian form, and let $(p(\delta), q(\delta))$, $p(\delta) + q(\delta) = m(\delta)$, be its signature. Define the signature character of $(\pi, V, \langle \cdot, \cdot \rangle)$ to be the formal pair of sums:

$$\Sigma(V) = \left(\sum_{\delta \in \widehat{\mathcal{W}}} p(\delta) \delta, \sum_{\delta \in \widehat{\mathcal{W}}} q(\delta) \delta \right). \quad (4.2.2)$$

It is clear that $(\pi, V, \langle \cdot, \cdot \rangle)$ is unitary if and only if $q(\delta) = 0$ for all $\delta \in \widehat{\mathcal{W}}$. The fundamental result that one needs relates the signature of an irreducible module to a combination of signatures of the tempered modules. In the original real groups setting, this is theorem 1.5 in [Vo]. In the setting of p -adic groups and affine Hecke algebras, this is theorem 5.2 in [BM1].

Theorem 4.2.3 ([Vo, BM1]). Let (π, V) be an irreducible \mathcal{K} -module having a nonzero hermitian form. Then there exist finitely many irreducible tempered modules (π_j, V_j) , and integers a_j^\pm , $j = 1, m$, such that

$$\Sigma(V) = \left(\sum_{j=1}^m a_j^+ \theta_{\mathcal{W}}(V_j), \sum_{j=1}^m a_j^- \theta_{\mathcal{W}}(V_j) \right).$$

Moreover, if (π, V) has real central character, then so do (π_j, V_j) , $j = 1, m$.

We remark that the integers a_j^\pm are not necessarily nonnegative, and that there is no known effective algorithm to compute them in our setting. The proof of the theorem is completely formal, once the results in section 4.1 are given. It uses the notion of Jantzen filtration, and an induction on the length of the Langlands parameter. One may simply follow the proof in [BM1].

4.3. We compare the signature characters for the Hecke algebras \mathcal{K} . The idea is due to [BM1, BM2]. Let $\bar{\chi}$ denote a central character for \mathcal{K} .

Definition 4.3.1. Let (π, V) be an irreducible \mathcal{K} -module with a nonzero hermitian form and corresponding signature character $\Sigma(V)$ as in theorem 4.2.3. We say that V has the Vogan property, if

$$\sum_{j=1}^m a_j^- \theta_{\mathcal{W}}(V_j) = 0 \text{ implies that } a_j^- = 0, \text{ for all } j = 1, m.$$

We say that a subcategory $\text{mod}_{\bar{\chi}}\mathcal{K}$ has the Vogan property if every irreducible module in this subcategory does.

Lemma 4.3.2. *Let \mathcal{K} be \mathbb{H} or \mathbb{H}' . Then the category of finite dimensional \mathcal{K} -modules with real central character has the Vogan property.*

Proof. This is immediate by proposition 3.5.1 and corollary 3.6.3, which say that the set of tempered \mathcal{K} -modules with real central character have linearly independent \mathcal{W} -characters. \square

We cannot apply the same argument for $\widetilde{\mathbb{H}}'$. The reason is the following. Note that, by corollary 2.6.2, in the correspondence (4.1.2), we have

$$\theta_{\widetilde{\mathcal{W}}'}(\widetilde{V}') = \text{Ind}_{\widetilde{\mathcal{W}}'}^{\widetilde{\mathcal{W}}'}(\theta_{\mathcal{W}'}(V')). \quad (4.3.1)$$

Consequently, it is possible to have distinct tempered modules with real central character for $\widetilde{\mathbb{H}}'$, which have the same $\widetilde{\mathcal{W}}'$ structure. In particular, the linear independence for tempered modules does not hold anymore. One bypasses this difficulty as in [BM2].

Proposition 4.3.1. *The category of finite dimensional $\widetilde{\mathbb{H}}'$ -modules with real central character has the Vogan property.*

Proof. Let \widetilde{V}' be an irreducible hermitian $\widetilde{\mathbb{H}}'$ -module with real central character. Assume that

$$\Sigma(\widetilde{V}') = \left(\sum_{j=1}^m a_j^+ \theta_{\widetilde{\mathcal{W}}'}(\widetilde{V}'_j), \sum_{j=1}^m a_j^- \theta_{\widetilde{\mathcal{W}}'}(\widetilde{V}'_j) \right).$$

The fact that the functor $V' \mapsto \widetilde{V}'$ in (4.1.2) is an isomorphism implies that

$$\Sigma(V') = \left(\sum_{j=1}^m a_j^+ \theta_{\mathcal{W}'}(V'_j), \sum_{j=1}^m a_j^- \theta_{\mathcal{W}'}(V'_j) \right).$$

By lemma 2.7.3, all V'_j are tempered with real central character. If $\sum_{j=1}^m a_j^- \theta_{\widetilde{\mathcal{W}}'}(\widetilde{V}'_j) = 0$, then \widetilde{V}' unitary, and so V' is unitary too. This means $\sum_{j=1}^m a_j^- \theta_{\mathcal{W}'}(V'_j) = 0$. The claim follows now by lemma 4.3.2. \square

4.4. The main application is next. Call an affine algebra $\mathcal{H}^{\lambda, \lambda^*}(\Psi)$ of geometric type, if all the graded algebras \mathbb{H}_{μ_α} that appear via the reductions in section 2 are of geometric type in the sense of section 3.

Theorem 4.4.1. *Assume the notation from section 2. Let $\mathcal{H}^{\lambda, \lambda^*}(\Psi, z_0)$ be the affine Hecke algebra attached to a root datum Ψ (definition 2.1.1) and with parameters λ, λ^* of geometric type (section 3), and assume z_0 is not a root of unity, and let $\mathcal{H}'(\Psi, z_0) = \mathcal{H}^{\lambda, \lambda^*}(\Psi, z_0) \rtimes \Gamma$ be an extended Hecke algebra. Let $s_e \in T_e$ be a fixed elliptic semisimple element. The category of finite dimensional $\mathcal{H}^{\lambda, \lambda^*}(\Psi, z_0)$ -modules whose central characters have elliptic parts $G(\Psi) \rtimes \Gamma$ -conjugate to s_e has the Vogan property (definition 4.3.1).*

Proof. This follows now from propositions 2.5.1, 2.6.1, and 4.3.1. \square

5. ELEMENTS OF THE THEORY OF TYPES

We need to recall certain elements of the theory of types (see [BK2] and the references therein).

5.1. Let \mathbb{F} denote a p -adic field of characteristic zero, with norm $\|\cdot\|$, and let \mathcal{G} be the group of \mathbb{F} -rational points of a connected, reductive, algebraic group defined over \mathbb{F} . Let $\mathfrak{R}(\mathcal{G})$ denote the category of smooth complex representations of \mathcal{G} .

A character $\chi : \mathcal{G} \rightarrow \mathbb{C}^\times$ is called unramified, if there exist \mathbb{F} -rational characters $\phi_j : \mathcal{G} \rightarrow \mathbb{F}^\times$, $j = 1, k$, and complex numbers s_j , $j = 1, k$, such that $\chi(g) = \prod_{j=1}^k \|\phi_j(g)\|^{s_j}$, for all $g \in \mathcal{G}$.

We recall Bernstein's decomposition of $\mathfrak{R}(\mathcal{G})$ adapted to the theory of types. One defines an equivalence relation on the set of pairs (L, σ) , where L is a \mathbb{F} -rational Levi subgroup of \mathcal{G} , and σ is an irreducible supercuspidal representation of L as follows.

Definition 5.1.1. Two pairs (L_1, σ_1) and (L_2, σ_2) are inertially equivalent if there exists $g \in \mathcal{G}$, and an unramified character χ of L_2 such that $gL_1g^{-1} = L_2$, and $g \cdot \sigma_1 = \sigma_2 \otimes \chi$.

Let $\mathfrak{B}(\mathcal{G})$ denote the set of inertial equivalence classes. If (π, V) is an irreducible representation in $\mathfrak{R}(\mathcal{G})$, then there exists a pair (P, σ) , where $P = LN$ is a \mathbb{F} -rational parabolic and σ is an irreducible supercuspidal representation of L , such that π is equivalent with a subquotient of the normalized induced representation $\text{Ind}_P^{\mathcal{G}}(\sigma)$. Moreover, the pair (L, σ) is unique up to conjugacy, so in particular, one can assign to π , in a well-defined way, the inertial class $\mathfrak{s} = [L, \sigma] \in \mathfrak{B}(\mathcal{G})$. This is called the inertial support of π . More generally, one defines a full subcategory $\mathfrak{R}^{\mathfrak{s}}(\mathcal{G})$ whose objects are (π, V) (not necessarily irreducible) for which all irreducible subquotients have inertial support \mathfrak{s} .

Remarks 5.1.2. (1) The inertial support gives a decomposition

$$\mathfrak{R}(\mathcal{G}) = \prod_{\mathfrak{s} \in \mathfrak{B}(\mathcal{G})} \mathfrak{R}^{\mathfrak{s}}(\mathcal{G}). \quad (5.1.1)$$

(2) This decomposition is well behaved with respect to Langlands classification. By a result of [Cas], if the Langlands subquotient of a standard module has inertial support \mathfrak{s} , then the standard module is in $\mathfrak{R}^{\mathfrak{s}}(\mathcal{G})$.

5.2. Let $\mathcal{H}(\mathcal{G})$ denote the Hecke algebra of \mathcal{G} , *i.e.* the space of locally constant, compactly supported complex functions on \mathcal{G} with convolution with respect to some fixed Haar measure. The algebra $\mathcal{H}(\mathcal{G})$ has a natural $*$ -operation: if $f \in \mathcal{H}(\mathcal{G})$, then $f^*(g) := \overline{f(g^{-1})}$. Let J be a compact open subgroup of \mathcal{G} , and fix a smooth irreducible representation (ρ, \mathcal{W}) of J . Let $(\rho^\vee, \mathcal{W}^\vee)$ be the contragredient representation.

Definition 5.2.1. The Hecke algebra $\mathcal{H}(\mathcal{G}, \rho)$ is the associative complex convolution algebra with unit on the space of compactly supported functions $f : \mathcal{G} \rightarrow \text{End}_{\mathbb{C}}(\mathcal{W}^\vee)$ satisfying

$$f(j_1 g j_2) = \rho^\vee(j_1) \circ f(g) \circ \rho^\vee(j_2), \quad j_1, j_2 \in J, g \in \mathcal{G}.$$

It has a natural $*$ -operation defined by $f^*(g) := \overline{f(g^{-1})}^\vee$, where $^\vee$ denotes the transpose with respect to the pairing between \mathcal{W} and \mathcal{W}^\vee .

There is a natural isomorphism $\mathcal{H}(\mathcal{G}, \rho) \cong \text{End}_G(\text{Ind}_J^G(\rho))$. On the other hand, there exists an idempotent element $e_\rho \in \mathcal{H}(\mathcal{G})$ defined using ρ as in [BK2], 2.9. Then one can consider the algebra $\mathcal{H}(G)_\rho := e_\rho \star \mathcal{H}(G) \star e_\rho$. There is a natural isomorphism (proposition 2.11 in [BK2])

$$\mathcal{H}(G, \rho) \otimes_{\mathbb{C}} \text{End}_{\mathbb{C}}(\mathcal{W}) \cong \mathcal{H}(G)_\rho. \quad (5.2.1)$$

If $(\pi, V) \in \mathfrak{R}(G)$, define

$$V_\rho := \text{Hom}_J[\mathcal{W}, V] \text{ and } V^\rho := \pi(e_\rho)V. \quad (5.2.2)$$

The first is the space of ρ -invariants in V , and the second turns out to be the ρ -isotypic component of V . Clearly, the second space acquires a $\mathcal{H}(\mathcal{G})_\rho$ -module structure. The first space is a left $\mathcal{H}(\mathcal{G}, \rho)$ -module as follows. If $\phi \in \text{Hom}_J[\mathcal{W}, V]$ and $f \in \mathcal{H}(\mathcal{G}, \rho)$, then $\pi(f)\phi$ is the homomorphism

$$\mathcal{W} \ni w \mapsto \int_G \pi(g) \phi((f(g))^\vee w) dg. \quad (5.2.3)$$

Let $\mathfrak{R}_\rho(\mathcal{G})$ denote the full subcategory of $\mathfrak{R}(\mathcal{G})$ whose objects are representations (π, V) such that V is generated by V^ρ , *i.e.* $V = \mathcal{H}(\mathcal{G}) \star V^\rho$. One has two functors therefore:

$$\mathbf{m}_\rho : \mathfrak{R}_\rho(\mathcal{G}) \rightarrow \mathcal{H}(\mathcal{G})_\rho - \text{mod}, \quad \mathbf{m}_\rho(V) = V^\rho, \quad (5.2.4)$$

$$\mathbf{M}_\rho : \mathfrak{R}_\rho(\mathcal{G}) \rightarrow \mathcal{H}(\mathcal{G}, \rho) - \text{mod}, \quad \mathbf{M}_\rho(V) = V_\rho. \quad (5.2.5)$$

5.3. We retain the notation from the previous sections.

Definition 5.3.1. The pair (J, ρ) is called a type in \mathcal{G} if the category $\mathfrak{R}_\rho(\mathcal{G})$ is closed under subquotients.

The main foundational result is next.

Theorem 5.3.2 ([BK2], 3.12, 4.3). *Assume that (J, ρ) is a type in \mathcal{G} . Then we have the following:*

- (i) *the functor \mathbf{m}_ρ is an equivalence of categories;*
- (ii) *the functor \mathbf{M}_ρ is an equivalence of categories;*
- (iii) *there exists a finite subset $\mathfrak{S} \subset \mathfrak{B}(\mathcal{G})$ such that*

$$\mathfrak{R}_\rho(\mathcal{G}) = \prod_{\mathfrak{s} \in \mathfrak{S}} \mathfrak{R}^{\mathfrak{s}}(\mathcal{G}). \quad (5.3.1)$$

In fact, these properties characterize the notion of type. When (5.3.1) holds, we call (J, ρ) a \mathfrak{S} -type.

5.4. We need a result from [BK2], section 11, which gives a Bernstein-type of decomposition for $\mathcal{H}(\mathcal{G}, \rho)$. Assume for simplicity from now on that (J, ρ) is a \mathfrak{s} -type, where $\mathfrak{s} = [L, \sigma] \in \mathfrak{B}(\mathcal{G})$. This also gives rise to an element $\mathfrak{s}_L = [L, \sigma] \in \mathfrak{B}(L)$, and a \mathfrak{s}_L -type (J_L, ρ_L) in L , such that conditions (11.1) in [BK2] are satisfied. Let us fix a maximal compact open subgroup \mathcal{K} of \mathcal{G} such that $J \subset \mathcal{K}$, and parabolic $P = LN$ satisfying hypothesis (11.2) in [BK2].

Theorem 5.4.1 ([BK2], 11.4). *There is a natural isomorphism of \mathbb{C} -vector spaces*

$$\delta_{\mathfrak{s}} : \mathcal{H}(K, \rho) \otimes_{\mathbb{C}} \mathcal{H}(L, \rho_L) \rightarrow \mathcal{H}(\mathcal{G}, \rho),$$

given by convolution. The algebra $\mathcal{H}(L, \rho_L)$ is abelian.

Using the $*$ -operation of $\mathcal{H}(G, \rho)$ we may define hermitian and unitary $\mathcal{H}(G, \rho)$ -modules.

Definition 5.4.2. The \mathfrak{s} -type (J, ρ) is called *affine* if there exists a $*$ -preserving isomorphism

$$\xi : \mathcal{H}(G, \rho) \rightarrow \mathcal{H}'(\Psi),$$

where $\mathcal{H}'(\psi)$ is an extended affine Hecke algebra from definition 2.1.1² of geometric type and such that it preserves the Bernstein decomposition, *i.e.* in the notation of theorem 5.4.1 and (2.2.2):

$$(\xi \circ \delta_{\mathfrak{s}})(\mathcal{H}(K, \rho)) = \mathcal{H}_{W'}, \quad (\xi \circ \delta_{\mathfrak{s}})(\mathcal{H}(L, \rho_L)) = \mathcal{A}(\Psi).$$

Moreover we require that the isomorphism ξ takes standard modules to standard modules, and in particular tempered modules to tempered modules.

Then the main observation is that under these hypothesis, since corollary 4.4.1 holds, the proof of theorem 1.1 in [BM1] can be repeated to obtain the following result. The necessary results relating parabolic induction and the equivalence $\mathbf{M}_{\rho} : \mathfrak{R}^{\mathfrak{s}}(\mathcal{G}) \cong \mathcal{H}(\mathcal{G}, \rho)\text{-mod}$ are in section 8 of [BK2].

Corollary 5.4.3. *Assume the \mathfrak{s} -type (J, ρ) is affine in the sense of definition 5.4.2. Then, in the equivalence of categories*

$$\mathbf{M}_{\rho} : \mathfrak{R}^{\mathfrak{s}}(\mathcal{G}) \rightarrow \mathcal{H}(\mathcal{G}, \rho)\text{-mod},$$

the hermitian and unitary modules correspond, respectively.

When the hermitian and unitary modules between two categories correspond functorially, we say that the two categories are *unitarily equivalent*.

6. UNITARY CORRESPONDENCES

In this section, we give two examples of unitary correspondences as in corollary 5.4.3, and present certain applications to unitary functorial correspondences with endoscopic groups.

²for some specialized value $z_0 \in \mathbb{R}_{>1}$ of the indeterminate z

6.1. Unramified principal series. Assume that \mathcal{G} is the \mathbb{F} -points of a linear reductive algebraic group over \mathbb{F} . Denote by $v_{\mathbb{F}}$ the valuation function on \mathbb{F} . Standard references for the discussion about unramified principal series are [Car], [Bo1].

Fix A a maximally split torus in \mathcal{G} and set

$$M = C_{\mathcal{G}}(A), \quad W(\mathcal{G}, A) = N_{\mathcal{G}}(A)/M. \quad (6.1.1)$$

Let $X^*(M)$ and $X_*(M)$ denote the lattices of algebraic characters and cocharacters of M , respectively, and $\langle \cdot, \cdot \rangle$ their natural pairing. Define the valuation map

$$\begin{aligned} v_M : M &\rightarrow X_*(M), \langle \lambda, v_M(m) \rangle = v_{\mathbb{F}}(\lambda(m)), \\ \forall m \in M, \lambda &\in X^*(M). \end{aligned} \quad (6.1.2)$$

Set ${}^0M = \ker v_M$ and $\Lambda(M) = \mathfrak{S}v_M$. Similarly, define v_A , 0A , and $\Lambda(A)$. Since A is a torus, we have $\Lambda(A) = X_*(A)$. Moreover, we have $X_*(A) \subset \Lambda(M) \subset X_*(M)$. (Notice that $\Lambda(M) = X_*(M)$ precisely when $M = A$, i.e., \mathcal{G} is \mathbb{F} -split.)

The group of unramified characters of M (i.e., characters trivial on 0M) will be denoted by \widehat{M}^u . For every character $\chi \in \widehat{M}^u$, let $X(\chi)$ denote the corresponding unramified principal series. It is clear that

$$\widehat{M}^u \cong \text{Hom}(\Lambda(M), \mathbb{C}^\times), \quad (6.1.3)$$

so if we define $T' = \text{Spec} \mathbb{C}[\Lambda(M)]$, a complex algebraic torus, we have a natural identification

$$\widehat{M}^u = T'. \quad (6.1.4)$$

Let \mathcal{K} and \mathcal{I} be a special maximal compact open subgroup of \mathcal{G} and an Iwahori subgroup, respectively, attached, using the Bruhat-Tits building, to the torus A and a special vertex x_0 (see [Ti], or section 3.5 in [Car]). Then ${}^0A = A \cap \mathcal{K}$ and ${}^0M = M \cap \mathcal{K}$. The Weyl group $W(\mathcal{G}, A)$ acts on $X_*(M)$ preserving $X_*(A)$ and $\Lambda(M)$. If we let $\widetilde{W}(\mathcal{G}, A) = W(\mathcal{G}, A) \rtimes \Lambda(M)$ denote the extended Weyl group, the Bruhat-Tits decomposition is

$$\mathcal{G} = \mathcal{I} \widetilde{W}(\mathcal{G}, A) \mathcal{I} \text{ and } \mathcal{K} = \mathcal{I} W(\mathcal{G}, A) \mathcal{I}. \quad (6.1.5)$$

The subquotients of the minimal (unramified) principal series $X(\chi)$ have inertial support $1 = [A, 1_A]$, where 1_A denotes the trivial character on A . In other words, the irreducible subquotients of the minimal principal series form the irreducible objects of the category $\mathfrak{R}^1(\mathcal{G})$.

Theorem 6.1.1 ([Bo1, Cas]). *The pair $(\mathcal{I}, 1_{\mathcal{I}})$ is a 1-type (in the sense of definition 5.3.1) for \mathcal{G} . In particular, $\mathfrak{R}^1(\mathcal{G}) = \mathfrak{R}_{(\mathcal{I}, 1_{\mathcal{I}})}(\mathcal{G})$, and tempered parameters match tempered parameters.*

In this case, the structure of $\mathcal{H}(\mathcal{G}, 1_{\mathcal{I}})$ is well-known by [IM]. Its description with generators and relations and the explicit parameters are in the tables of [Ti]. In the terminology of section 2, it is an affine Hecke algebra $\mathcal{H}(\Psi)$ with certain unequal parameters of geometric type for a root datum Ψ . More precisely, with the notation from section 2, particularly definition 2.1.1, we have $\Psi = (X, X^\vee, R, R^\vee)$, where

- (1) $X = X^*(T') (= \Lambda(M))$, $X^\vee = X_*(T')$;
- (2) the Weyl group of Ψ is $W(\mathcal{G}, A)$;
- (3) R^\vee is the set of ‘‘restricted roots’’ of A in \mathcal{G} (see [Ti], section 1.9, or [Car], page 141).

This implies that $(\mathcal{I}, 1_{\mathcal{I}})$ is an affine 1-type in the sense of definition 5.4.2, so we have:

Theorem 6.1.2. *The categories $\mathfrak{R}^1(\mathcal{G})$ and $\mathcal{H}(\mathcal{G}, 1_{\mathcal{I}})$ are unitarily equivalent.*

Remark 6.1.3. Following theorem 2.2.1, we see that the central characters of $\mathcal{H}(\mathcal{G}, 1_{\mathcal{I}})$ are in one-to-one correspondence with $W(\mathcal{G}, A)$ -conjugacy classes in T' . We will use this fact in section 6.2.

6.2. Quasisplit groups. We retain the notation from 6.1. In this subsection we explain a correspondence between unitarizable principal series of a quasisplit, nonsplit, quasisimple p -adic group \mathcal{G} and certain endoscopic split groups. We assume in addition that \mathcal{G} splits over an unramified extension of \mathbb{F} . The key observation is that while $\mathcal{H}(\mathcal{G}, 1_{\mathcal{I}})$ may have unequal parameters, all of the graded Hecke algebras attached can be identified naturally with graded Hecke algebras with equal parameters. For this we need to examine the Iwahori-Hecke algebra and its graded versions more closely. Assume that the root datum Ψ for $\mathcal{H}(\mathcal{G}, 1_{\mathcal{I}})$ is non-simply laced root datum, and with parameters λ and λ^* , the latter occurring when Ψ is of type B .

Let α_s, α_ℓ denote a short root and a long root respectively, and let $|\alpha|$ denote the length of a root α . The following lemma can be verified by inspecting Tits' tables for quasisplit groups ([Ti]).

Lemma 6.2.1. *The parameters of $\mathcal{H}(\mathcal{G}, 1_{\mathcal{I}})$ satisfy the conditions:*

- (1) $\frac{\lambda(\alpha_s)}{\lambda(\alpha_\ell)} \in \left\{1, \frac{|\alpha_s|}{|\alpha_\ell|}\right\};$
- (2) $\frac{\lambda(\alpha_s) \pm \lambda^*(\alpha_s)}{2\lambda(\alpha_\ell)} \in \left\{0, \frac{|\alpha_s|}{|\alpha_\ell|}, 1\right\}.$

These conditions guarantee that for every graded Hecke algebra \mathbb{H}_{μ_σ} that appears in remark 2.3.3 (via theorem 2.3.2), the parameters $\mu_\sigma(\alpha)$ satisfy one of the following properties:

- (a) $\mu_\sigma(\alpha_s) = \mu_\sigma(\alpha_\ell),$
- (b) $\mu_\sigma(\alpha_s) = \frac{|\alpha_s|}{|\alpha_\ell|} \mu_\sigma(\alpha_\ell),$
- (c) $\mu_\sigma(\alpha_s) = 0.$

In case (a), \mathbb{H}_{μ_σ} is a graded Hecke algebra with equal parameters. In case (b), we have a natural isomorphism

$$\mathbb{H}_{\mu_\sigma} \cong \mathbb{H}_{\mu_\sigma^\vee}, \quad (6.2.1)$$

where $\mathbb{H}_{\mu_\sigma^\vee}$ is the graded Hecke algebra attached to the dual root system to that of \mathbb{H}_{μ_σ} and with parameters $\mu_\sigma^\vee(\check{\alpha}_\ell) = \mu_\sigma(\alpha_\ell), \mu_\sigma^\vee(\check{\alpha}_s) = \mu_\sigma(\alpha_s)$. Notice that $\mathbb{H}_{\mu_\sigma^\vee}$ is a graded Hecke algebra with equal parameters. In case (c), we have a natural isomorphism (see proposition 4.6 in [BC])

$$\mathbb{H}_{\mu_\sigma} \cong \mathbb{C}[W_s] \times \mathbb{H}_{\mu_\sigma}^0, \quad (6.2.2)$$

where W_s is the reflection subgroup of W generated by the simple short roots, and $\mathbb{H}_{\mu_\sigma}^0$ is the graded Hecke algebra (with equal parameter $\mu_\sigma(\alpha_\ell)$) corresponding to the root system of long roots.

Every unramified principal series $X(\chi)$ contains a unique irreducible (\mathcal{K} -)spherical subquotient $\overline{X}(\chi)$. It is well-known that two unramified principal series $X(\chi)$ and $X(\chi')$ have the same composition factors, and in particular $\overline{X}(\chi) \cong \overline{X}(\chi')$ if and only if

$$\chi = w\chi', \text{ for some } w \in W(\mathcal{G}, A).$$

Let $\widehat{\mathcal{G}}_{\text{sph}}$ denote the set of isomorphism classes of irreducible spherical representations of \mathcal{G} . Therefore, we have a one-to-one correspondence

$$\widehat{\mathcal{G}}_{\text{sph}} \longleftrightarrow W(\mathcal{G}, A)\text{-orbits in } \widehat{M}^u = T'. \quad (6.2.3)$$

(A better way to express this correspondence is via the Satake isomorphism $\mathcal{H}(\mathcal{G}, \mathcal{K}) \cong \mathbb{C}[\widehat{M/0M}]^{W(\mathcal{G}, A)}$, see for example theorem 4.1 in [Car].)

We need to recast this bijection in terms of the dual L-group. Let G denote the complex connected group dual (in the sense of Langlands) to \mathcal{G} , and let T be a maximal torus in G . Let $\Psi(G) = (X^*(T), X_*(T), R(G, T), R^\vee(G, T))$ be the corresponding root datum. The inner class of \mathcal{G} defines a homomorphism $\tau : \Gamma \rightarrow \text{Aut}(\Psi(G))$, where $\Gamma = \text{Gal}(\overline{\mathbb{F}}/\mathbb{F})$. Since we assumed that \mathcal{G} is quasisimple unramified, we know that the image $\tau(\Gamma) \subset \text{Aut}(\Psi(G))$ is a cyclic group generated by an automorphism of order d ($d \in \{2, 3\}$) which, by abuse of notation, we also call τ . We fix a choice of root vectors X_α , for $\alpha \in R$. The automorphism τ maps the root space of α to the root space of $\tau(\alpha)$. We normalize τ such that

$$\tau(X_\alpha) = X_{\tau(\alpha)}, \text{ for all } \alpha. \quad (6.2.4)$$

This allows one to extend τ to an automorphism of G in a canonical way.

Definition 6.2.2. Two elements $x_1, x_2 \in G$ are called τ -conjugate if there exists $g \in G$ such that $x_2 = g \cdot x_1 \cdot \tau(g^{-1})$. For a subset $S \subset G$, denote:

$$N_G(S\tau) = \{g \in G : g \cdot S \cdot \tau(g^{-1}) \subset S\}.$$

The construction of the L-group is such that we have (see [Bo2], section 6):

$$X^*(T') = X^*(T)^\tau, \quad W(\mathcal{G}, A) = W(G, T)^\tau. \quad (6.2.5)$$

In particular, we have an inclusion $X^*(T') \hookrightarrow X^*(T)$, which gives a surjection $\nu : T \rightarrow T'$. By lemma 6.4 in [Bo2], the map

$$\nu' : T \rtimes \langle \tau \rangle \rightarrow T', \quad \nu'((t, \tau)) = \nu(t), \quad (6.2.6)$$

induces a bijection of $(N(T\tau), \tau)$ -conjugacy classes of elements in T and $W(G, T)^\tau$ -conjugacy classes of elements in T' .

Theorem 6.2.3 (Langlands, cf. [Car], theorem 3.1, [Bo2], proposition 6.7). *There are bijective correspondences:*

$$\begin{aligned} \text{semisimple } \tau\text{-conjugacy classes in } G & \longleftrightarrow \\ (N_G(T\tau), \tau)\text{-conjugacy classes of elements in } T & \longleftrightarrow \\ W(G, T)^\tau\text{-orbits in } T' (\longleftrightarrow \widehat{\mathcal{G}}_{sph}). & \end{aligned} \quad (6.2.7)$$

Definition 6.2.4. Let π be an irreducible representation in $\mathfrak{R}^1(\mathcal{G})$. It occurs as a subquotient in an unramified principal series $X(\chi)$ with \mathcal{K} -spherical subquotient $\overline{X}(\chi)$. Via theorem 6.2.3, to $\overline{X}(\chi)$ there corresponds a semisimple τ -conjugacy class in G . We will refer to this class (or any representative of it) as the infinitesimal character of π .

Notice that definition 6.2.4 is compatible, via remark 6.1.3, to our conventions from section 2.

Definition 6.2.5. We fix now an elliptic element $s_e \in T^\tau$, and consider the subcategory $\mathfrak{R}_{s_e}^1(\mathcal{G}) \subset \mathfrak{R}^1(\mathcal{G})$ of representations of \mathcal{G} with infinitesimal character having elliptic part τ -conjugate to s_e .

Let

$$G(s_e\tau) = \{g \in G : gs_e\tau(g^{-1}) = s_e\} \quad (6.2.8)$$

denote the twisted centralizer of s_e in G . This is a potentially disconnected reductive group. Denote the identity component by $G(s_e\tau)_0$. Let $\mathcal{G}(s_e\tau)$ denote a split p -adic group whose

TABLE 1. Nonsplit quasisplit unramified \mathcal{G} and corresponding split $\mathcal{G}(\tau)$.

Type of \mathcal{G}	Name in [Ti]	Order of τ	Type of $\mathcal{G}(\tau)$
A_{2n-1}	${}^2A'_{2n-1}$	2	B_n
A_{2n}	${}^2A'_{2n}$	2	C_n
D_n	2D_n	2	C_{n-1}
D_4	3D_4	3	G_2
E_6	2E_6	2	F_4

Langlands dual is $G(s_e\tau)$. In particular, this means that the group is the split \mathbb{F} -points of a disconnected group whose identity component has root datum dual (in the sense of Langlands) to that of $G(s_e\tau)_0$, and whose group of components is isomorphic to $G(s_e\tau)/G(s_e\tau)_0$.

Using the analysis after Lemma 6.2.1, based on a case-by-case inspection of the tables in [Ti], we find that the graded algebra at s_e of the affine Iwahori-Hecke algebras for \mathcal{G} is naturally $*$ -preserving isomorphic with the graded algebra at 1 of $\mathcal{G}(s_e\tau)$. When $s_e = 1$, we denote this group by $\mathcal{G}(\tau)$. We obtain the following consequence.

Corollary 6.2.6. *The categories $\mathfrak{R}_{s_e}^1(\mathcal{G})$ and $\mathfrak{R}_1^1(\mathcal{G}(s_e\tau))$ are unitarily equivalent. In particular, there is a unitary equivalence for representations with Iwahori fixed vectors and real infinitesimal character between \mathcal{G} and $\mathcal{G}(\tau)$.*

Example 6.2.7. The explicit cases for quasisplit groups and real infinitesimal characters are in table 6.2. For example, when \mathcal{G} is the quasisplit adjoint unitary group in four variables ($\mathcal{G} = PSU(2, 2)$), we have $G = SL(4, \mathbb{C})$. The corresponding automorphism τ is

$$\tau(g) = J \cdot g \cdot J^t, \text{ where } J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \quad (6.2.9)$$

The torus T is the diagonal one. If we choose $s_e = 1$, then $G(\tau) = Sp(4, \mathbb{C})$, so the correspondence of unitarity is with $\mathcal{G}(\tau) = SO(5, \mathbb{F})$, but if we choose $s_e = \text{diagonal}(1, 1, -1, -1)$, then $G(s_e\tau) = SO(4, \mathbb{C})$, and the correspondence of unitarity is with $\mathcal{G}(s_e\tau) = SO(4, \mathbb{F})$.

6.3. Ramified principal series for split groups. A similar type of correspondence can be achieved for ramified principal series using the results of [Ro]. We assume that \mathcal{G} is split over \mathbb{F} and the same restrictions of the characteristic p as in [Ro]. Let $\chi : A \rightarrow \mathbb{C}^\times$ be a ramified character and set ${}^0\chi : {}^0A \rightarrow \mathbb{C}^\times$, ${}^0\chi = \chi|_{{}^0A}$. One considers the inertial class $\mathfrak{r} = [A, \chi]$. This only depends on ${}^0\chi$. The irreducible objects in $\mathfrak{R}^{\mathfrak{r}}(\mathcal{G})$ are the irreducible subquotients of minimal principal series $X(\chi')$, where ${}^0\chi' = {}^0\chi$.

In [Ro], a \mathfrak{r} -type (J, ρ) is constructed, with ρ one dimensional, and the structure of the Hecke algebras is computed. Let G be the complex group dual to \mathcal{G} . In [Ro], a semisimple element $\hat{{}^0\chi} \in G$ is attached to ${}^0\chi$. We explain this construction next. Let $\mathcal{W}_{\mathbb{F}}$ be the Weil group and recall the short exact sequence

$$1 \longrightarrow I_{\mathbb{F}} \longrightarrow \mathcal{W}_{\mathbb{F}} \longrightarrow \mathbb{Z} \longrightarrow 1, \quad (6.3.1)$$

where $I_{\mathbb{F}}$ is the inertia group (open in $\mathcal{W}_{\mathbb{F}}$), and the cyclic group \mathbb{Z} generated by a Frobenius element $\text{Frob} : x \mapsto x^q$ in $\text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$. The Weil-Deligne group is $\mathcal{W}'_{\mathbb{F}} = \mathbb{C} \rtimes \mathcal{W}_{\mathbb{F}}$, where $w \in \mathcal{W}_{\mathbb{F}}$ acts on \mathbb{C} by multiplication by $\|w\|$, the norm of w . In particular, Frob acts by multiplication by q .

A Weil homomorphism $\phi : \mathcal{W}'_{\mathbb{F}} \rightarrow G$ is a continuous homomorphism satisfying certain properties (see for example [Bo2], §8.1). In particular, $\phi|_{\mathcal{W}_{\mathbb{F}}}$ should consist of semisimple elements and $\phi|_{\mathbb{C}}$ of unipotent elements. We say that ϕ is unramified if $\phi(I_{\mathbb{F}})$ consists of central elements of G .

Let $\mathcal{W}_{\mathbb{F}}^{\text{ab}}$ denote the abelian quotient of $\mathcal{W}_{\mathbb{F}}$. The Weil homomorphisms that parameterize minimal principal series descend to $\mathbb{C} \times \mathcal{W}_{\mathbb{F}}^{\text{ab}}$. Let $I_{\mathbb{F}}^{\text{ab}}$ denote the image of $I_{\mathbb{F}}$ in $\mathcal{W}_{\mathbb{F}}^{\text{ab}}$. Recall that the reciprocity homomorphism $\tau_{\mathbb{F}}$ is an isomorphism of $\mathcal{W}_{\mathbb{F}}^{\text{ab}}$ onto \mathbb{F}^{\times} , and induces an isomorphism of $I_{\mathbb{F}}^{\text{ab}}$ onto \mathbb{O}^{\times} .

To the character ${}^0\chi : {}^0A \rightarrow \mathbb{C}^{\times}$, we attach a homomorphism

$$\widehat{{}^0\chi} : I_{\mathbb{F}}^{\text{ab}} \rightarrow T \subset G. \quad (6.3.2)$$

This is the unique homomorphism which makes the following diagram commutative for any $\lambda \in X_*(A) = X^*(T)$:

$$\begin{array}{ccc} I_{\mathbb{F}}^{\text{ab}} & \xrightarrow{\widehat{{}^0\chi}} & T \\ \tau_{\mathbb{F}} \downarrow & & \downarrow \lambda \\ \mathbb{O}^{\times} & & \mathbb{C}^{\times} \\ & \searrow \lambda & \nearrow {}^0\chi \\ & & {}^0A \end{array} \quad (6.3.3)$$

Now we consider

$$\phi : \mathbb{C} \times \mathcal{W}_{\mathbb{F}}^{\text{ab}} \rightarrow G \text{ such that } \phi|_{I_{\mathbb{F}}^{\text{ab}}} = \widehat{{}^0\chi}. \quad (6.3.4)$$

Such homomorphisms parameterize L-packets which have subquotients of minimal principal series $X(\chi)$, with $\chi|_{{}^0A} = {}^0\chi$. Define

$$C_G(\widehat{{}^0\chi}) = \text{centralizer in } G \text{ of the image of } \widehat{{}^0\chi}. \quad (6.3.5)$$

Lemma 6.3.1 ([Ro]). $C_G(\widehat{{}^0\chi})$ is the centralizer of a single semisimple element in G .

By abuse of notation, we denote this semisimple element by $\widehat{{}^0\chi}$ too. Let $\Psi_{\widehat{{}^0\chi}}$ be the root datum for the identity component of the centralizer $C_G(\widehat{{}^0\chi})$, and let Γ be the group of components of $C_G(\widehat{{}^0\chi})$.

Theorem 6.3.2 ([Ro]). The \mathfrak{r} -type (J, ρ) is affine, and the Hecke algebra $\mathcal{H}(G, \rho)$ is naturally isomorphic to the equal parameter extended affine Hecke algebra $\mathcal{H}'(\Psi_{\widehat{{}^0\chi}}) = \mathcal{H}(\Psi_{\widehat{{}^0\chi}}) \rtimes \Gamma$.

Corollary 6.3.3. The categories $\mathfrak{R}^{\mathfrak{r}}(\mathcal{G})$ and $\mathcal{H}(\mathcal{G}, \rho)$ -mod (equivalently, $\mathcal{H}'(\Psi_{\widehat{{}^0\chi}}) = \mathcal{H}(\Psi_{\widehat{{}^0\chi}}) \rtimes \Gamma$ -mod) are unitarily equivalent.

By combining this with theorem 6.1.2, we have an important consequence. Let $\mathcal{G}'({}^0\chi)$ be a split p -adic group dual to $C_G({}^0\chi)$. The Iwahori Hecke algebra of this group is naturally identified with $\mathcal{H}'(\Psi_{\widehat{{}^0\chi}})$. By combining the previous corollary with theorem 6.1.2, we have an important consequence.

Corollary 6.3.4. The categories $\mathfrak{R}^{\mathfrak{r}}(\mathcal{G})$ and $\mathfrak{R}^{\circ}(\mathcal{G}'({}^0\chi))$ are unitarily equivalent via the equivalences of categories:

$$\mathfrak{R}^{\mathfrak{r}}(\mathcal{G}) \xrightarrow{\cong} \mathcal{H}'(\Psi_{\widehat{{}^0\chi}})\text{-mod} \xleftarrow{\cong} \mathfrak{R}^{\circ}(\mathcal{G}'({}^0\chi)).$$

Assume now that \mathcal{G} is adjoint. In this setting, [Re] gives the Deligne-Langlands-Lusztig classification for $\mathfrak{R}^{\text{f}}(\mathcal{G})$. Notice that any parameter ϕ as in (6.3.4) has the image in fact in $G(\widehat{0\chi})$. Denote by ϕ' the homomorphism obtained by restricting the range of ϕ to $G(\widehat{0\chi})$. Then ϕ' is, by definition, an unramified parameter for $\mathcal{G}(0\chi)$. Therefore, we have a one-to-one correspondence of L-packets

$$\Psi : \phi \longrightarrow \phi', \quad \phi|_{I_{\mathbb{F}}^{\text{ab}}} = \widehat{0\chi}, \quad \phi' \text{ unramified for } \mathcal{G}(0\chi). \quad (6.3.6)$$

The assumption that \mathcal{G} is adjoint implies that $\mathcal{G}(0\chi)$ is connected. Following [Re], the correspondence (6.3.6) encodes the bijection between subquotients for the 0χ -ramified principal series of $G(\mathbb{F})$ and subquotients ϕ' of the unramified principal series of $G(0\chi)$. More precisely, let $A_G(\phi)$ and $A_{G(\widehat{0\chi})}(\phi')$ denote the component groups of the centralizers of the images of ϕ and ϕ' respectively. Let $\mathcal{B}_{G(\widehat{0\chi})}^{\phi}$ and $\mathcal{B}_{G(\widehat{0\chi})}^{\phi'}$ denote the varieties of Borel subgroups fixed by the images of ϕ and ϕ' respectively. We say that a representation of the component group is of Springer type if it appears in the action on the Borel-Moore homology of these varieties. Then there is a natural isomorphism

$$A_G(\phi) \cong A_{G(\widehat{0\chi})}(\phi'), \quad (6.3.7)$$

which induces a bijection Ψ of the component group representations of Springer type.

Then the reformulation of the corollaries in section 6.3, in the particular case when \mathcal{G} is adjoint, is that the correspondence Ψ of (6.3.6) restricted to elements of Springer type, gives a one-to-one correspondence between hermitian and unitary representations, respectively.

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