UNITARY EQUIVALENCES FOR REDUCTIVE $p$–ADIC GROUPS

DAN BARBASCH AND DAN CIUBOTARU

Abstract. We establish a transfer of unitarity for a Bernstein component of the category of smooth representations of a reductive $p$-adic group to the associated Hecke algebra, in the framework of the theory of types, whenever the Hecke algebra is an affine Hecke algebra with geometric parameters, in the sense of Lusztig (possibly extended by a group of automorphisms of the root datum). It is known that there is a large class of such examples (detailed in the paper). As a consequence, we establish relations between the unitary duals of different groups, in the spirit of endoscopy.

Contents

1. Introduction 1
2. Affine Hecke algebras and graded affine Hecke algebras 5
3. Geometric graded Hecke algebras and linear independence 13
4. Signature characters 18
5. Elements of the theory of types 23
6. Unitary correspondences 26
References 31

1. Introduction

1.1. A central result in the representation theory of $p$–adic groups is the Borel-Casselman equivalence [Bo1] between the category of smooth representations with Iwahori fixed vectors, and the category of modules of the Iwahori-Hecke algebra. This result has served as a paradigm for subsequent efforts to solve the problem of classifying the smooth dual of a $p$–adic group, [HM], [BK1], [K], [Ro], [Se] and many others. From the outset it was conjectured that the Borel-Casselman correspondence also preserves unitarity. In [BM1], it is proved that, for a split group $G$ of adjoint type over a $p$-adic field $F$, the Borel-Casselman equivalence between the categories of representations with Iwahori fixed vectors, and finite dimensional modules of the affine Hecke algebra preserves unitarity. The results depend heavily on the fact that the affine Hecke algebras has equal parameters, and corresponds to an adjoint group. In addition, one has to make a certain technical assumption, namely that the infinitesimal character be real (cf. 1.4). This assumption is removed in [BM2].

The first purpose of this paper is to establish the preservation of unitarity in a much more general setting. To do this, we treat the case of an extended affine Hecke algebra with general parameters of geometric type, and coming from a group of arbitrary isogeny class. The work of [BM1] relies on combining the classification of representations of the affine Hecke algebra and the idea of signature character in [Vo]. In order to treat the larger class of Hecke
algebras in this paper, we have to analyze the relation between the affine Hecke algebra and the graded affine Hecke algebra as in [BM2], and also rely heavily on the classification of irreducible modules of the graded affine Hecke algebra in ([Lu1, Lu2, Lu3, Lu4]).

The main technical result that we prove in this direction is summarized in the following theorem. Let $\Psi = (X, X^\vee, R, R^\vee)$ be a root datum corresponding to a complex connected reductive group $G(\Psi)$. Let $\mathcal{H}^{\lambda, \lambda^*}(\Psi, z_0)$ be the affine Hecke algebra attached to $\Psi$ (Definition 2.1.1) and with parameters $\lambda, \lambda^*$ of geometric type (section 3). We assume that $z_0$ is not a root of unity. Let $\mathcal{H}'(\Psi, z_0) = \mathcal{H}^{\lambda, \lambda^*}(\Psi, z_0) \rtimes \Gamma$ be an extended affine Hecke algebra by a finite group $\Gamma$ acting by automorphisms of $\Psi$ (Definition 2.1.1).

Let $s_e \in T_e := X^\vee \otimes_{\mathbb{Z}} S^1$ be a fixed elliptic semisimple element in $G(\Psi)$.

**Theorem 1.1.1.** The category of finite dimensional $\mathcal{H}'(\Psi, z_0)$–modules whose central characters have elliptic parts $G(\Psi) \rtimes \Gamma$-conjugate to $s_e$ has the Vogan property (Definition 4.3.1).

1.2. Theorem 1.1.1 plays the same role as its counterparts in [BM1] and [BM2]; namely it provides a means to conclude that whenever a finite dimensional representation of the Hecke algebra is unitary, then the corresponding infinite dimensional admissible representation in the Bernstein component is also unitary. This is the content of Theorem 1.2.1 below. The details are in section 5.

Let $(J, \rho)$ be a type for the element $s = [L, \sigma]$ in the Bernstein center ($L$ a Levi subgroup, $\sigma$ a supercuspidal representation of $L$). Let $s_L$ be the Bernstein class for $L$ defined by $(L, \sigma)$ and assume that $s_L$ has a type $(J_L, \rho_L)$ such that $(J, \rho)$ is a cover of $(J_L, \rho_L)$ in the sense of [BK2, Definition 8.1]. Let $\mathcal{H}(G, \rho)$ and $\mathcal{H}(L, \rho_L)$ be the Hecke algebras (Definition 5.2.1). Let $P$ be a parabolic subgroup with Levi subgroup $L$. Let $t_P : \mathcal{H}(L, \rho_L) \to \mathcal{H}(G, \rho)$ be the embedding defined in [BK2, Corollary 7.12].

**Theorem 1.2.1.** Let $s = [L, \sigma]$ be an element in the Bernstein center of $G$ and $\mathfrak{R}^s(G)$ the corresponding Bernstein component. Assume that there exists a type $(J, \rho)$ (Definition 5.3.1) for $\mathfrak{R}^s(G)$ which satisfies the following conditions$^1$:

(I) the algebra $\mathcal{H}(G, \rho)$ is isomorphic to an extended affine Hecke algebra $\mathcal{H}'(\Psi, z_0)$ with geometric parameters, such that the isomorphism preserves the natural Hilbert algebra structures (as defined in section 2.6 and Definition 5.2.1);

(II) there exists a compact open subgroup $\mathcal{K} \supset J$ of $G$ with $G = \mathcal{K}P$, such that the isomorphism in (I) maps the Hecke algebra $\mathcal{H}(\mathcal{K}, \rho)$ to the finite Hecke subalgebra $\mathcal{H}_W$ of $\mathcal{H}'(\Psi, z_0)$ (as defined in section 2.2).

(III) the isomorphism in (I) maps $t_P(\mathcal{H}(L, \rho_L))$ onto $\mathcal{A}(\Psi)$ (as defined in section 2.2).

Then the equivalence of categories $M_\rho : \mathfrak{R}^s(G) \to \mathcal{H}(G, \rho)$-mod given by the theory of types (see (5.3.1)) induces a bijection between the irreducible unitary modules in $\mathfrak{R}^s(G)$ and $\mathcal{H}(G, \rho)$-mod.

Conditions (I)-(III), and thus Theorem 1.2.1, are expected to hold in great generality. They are known to hold at least in the following cases:

1. unramified principal series ([Bo1],[IM],[Ti]);
2. $GL(n, \mathbb{F})$ ([BK1]) and $GL(n, D)$ where $D$ is a division algebra over $\mathbb{F}$ ([Se]);
3. unipotent representations of simple groups of adjoint type ([Lu6, Lu7, Lu4]);
4. ramified principal series of split groups ([Ro]);

$^1$In practice, when one calculates the isomorphism in (I), the compact subgroup $\mathcal{K}$ in (II) is implicitly determined, and condition (III) is verified.
(5) pseudo-spherical principal series for double covers of split groups of simply-connected type ([LS]).

An important consequence of this approach is that it gives a unified framework for obtaining unitary categorical equivalences between two Bernstein components for possibly different groups. Whenever two Bernstein components are controlled by isomorphic affine Hecke algebras such that the conditions (I)-(III) above hold, one has a unitary equivalence. The example we present in detail to illustrate this is the case of ramified principal series of a split group where we obtain a correspondence with the unramified principal series of a (split) endoscopic group (Theorem 1.5.1).

There is another, subtler phenomenon: there are important cases when the two affine Hecke algebras are not isomorphic, but certain of their graded versions (in the sense of [Lu1]) are. Again our methods allow us to conclude similar unitary correspondences. The example that we present in detail is that of unramified principal series of quasisplit unramified groups, and the equivalences are again with unramified principal series of certain (split) endoscopic groups (Theorem 1.4.1). We explain these two examples in more detail in the introduction, after presenting the fundamental case of the Borel-Casselman equivalence.

1.3. Let $\mathbb{F}$ be a $p$-adic field of characteristic zero. Let $\mathcal{G}$ be the $\mathbb{F}$-points of a connected linear algebraic reductive group defined over $\mathbb{F}$. Let $A$ be a maximally split torus of $\mathcal{G}$ and set $M = C_G(A)$, the centralizer of $A$ in $\mathcal{G}$. Fix a special maximal compact open subgroup $K$ and an Iwahori subgroup $I \subset K$ of $\mathcal{G}$, attached by Bruhat-Tits theory to $A$. Let $0M = M \cap K$ be the maximal compact open subgroup of $M$. A complex character $\chi : M \rightarrow \mathbb{C}^\times$ is called unramified if $\chi|_{0M} = 1$, and it is called ramified otherwise.

Fix $0\chi$ a character of $0M$, and consider the category $\mathcal{R}^{0\chi}(\mathcal{G})$ of smooth $\mathcal{G}$-representations, which appear as constituents of the minimal principal series induced from complex characters $\chi$ of $M$ such that $\chi|_{0M} = 0\chi$. This category is a Bernstein component of the category of all smooth $\mathcal{G}$-representations. We are interested in the study of hermitian and unitary modules in $\mathcal{R}^{0\chi}(\mathcal{G})$.

The first case is when $0\chi = 1$, the unramified principal series. In this case, the category $\mathcal{R}^1(\mathcal{G})$ is known to be naturally equivalent with the category of modules over the Iwahori-Hecke algebra $\mathcal{H}(\mathcal{G}, 1_I)$ ([Bo1]). The algebra $\mathcal{H}(\mathcal{G}, 1_I)$ is an affine Hecke algebra, possibly with unequal parameters (Definition (2.1.1)), and has a natural $*$-operation (section 2.6); therefore one can define hermitian and unitary modules.

**Theorem 1.3.1.** In the Borel-Casselman equivalence $\mathcal{R}^1(\mathcal{G}) \cong \mathcal{H}(\mathcal{G}, 1_I)$-mod, the hermitian and unitary representations correspond, respectively.

In this theorem, the group $\mathcal{G}$ is of arbitrary isogeny and not necessarily split, and we emphasize that the correspondence is via the equivalence of categories. The case when $\mathcal{G}$ is adjoint and split over $\mathbb{F}$ is in [BM1, BM2]. In order to prove the claim, we need to extend the methods of Barbasch-Moy so that we cover extended affine (and affine graded in the sense of [Lu1]) Hecke algebras with unequal parameters.

1.4. Here is a first important consequence of our methods, indicative of endoscopy. Assume that $\mathcal{G}$ is quasisplit quasisimple, and that it splits over an unramified extension of $\mathbb{F}$, and $K$ is hyperspecial. (When $\mathcal{G}$ is simple of adjoint type, the Deligne-Langlands-Lusztig classification for the representations in $\mathcal{R}^1(\mathcal{G})$ (and more generally, for unipotent representations) is in [Lu7].) We would like to relate the unitary representations in $\mathcal{R}^1(\mathcal{G})$ with the unitary
dual of certain split endoscopic groups. The Langlands complex dual group \( G \) is equipped with an automorphism \( \tau \) of the root datum of \( G \), defined by the inner class of \( \mathcal{G} \), see section 6.2. If \( \mathcal{G} \) is in fact split, then \( \tau \) is trivial. It is well-known that \( W(\mathcal{G}, A) \)-conjugacy classes of unramified complex characters of \( M \) are in one-to-one correspondence with \( \tau \)-twisted semisimple conjugacy classes in \( G \). In this correspondence, if \( X \) is a subquotient of a principal series induced from an unramified character \( \chi \), we refer to the corresponding twisted semisimple conjugacy class in \( G \) as the infinitesimal character of \( X \). We say that \( X \) has real infinitesimal character if the corresponding semisimple class is hyperbolic modulo the center.

Fix a semisimple elliptic element \( s_e \in G \). Let \( G(s_e \tau) \) denote the centralizer of \( s_e \) in \( G \) under \( \tau \)-twisted conjugacy, a reductive group. When \( G \) is simply-connected (so \( \mathcal{G} \) is adjoint), \( G(s_e \tau) \) is a connected group, but not in general. Let \( \mathcal{G}(s_e \tau) \) denote the split \( \mathbb{F} \)-form of a (possibly disconnected) group dual to \( G(s_e \tau) \). There is a natural one-to-one correspondence between infinitesimal characters of \( \mathcal{G} \) with elliptic part \( \tau \)-conjugate to \( s_e \) and real infinitesimal characters of \( \mathcal{G}(s_e \tau) \). Moreover, we find that the affine graded at \( s_e \) Hecke algebras for \( \mathcal{G} \) and the affine graded at the identity Hecke algebra \( \mathcal{G}(s_e \tau) \) are naturally isomorphic. The methods of this paper then imply the following result.

**Theorem 1.4.1.** Let \( \mathcal{G} \) be a quasisplit quasisimple group which splits over an unramified extension of \( \mathbb{F} \). Fix \( s_e \) an elliptic element in the dual complex group \( G \). There is a natural one-to-one correspondence between irreducible representations in \( \mathcal{R}^1(\mathcal{G}) \) whose infinitesimal character has elliptic part \( \tau \)-conjugate with \( s_e \), and representations in \( \mathcal{R}^1(\mathcal{G}(s_e \tau)) \) with real infinitesimal character, such that the hermitian and unitary modules correspond, respectively.

For example, if \( s_e = 1 \) and \( \mathcal{G} \) is the quasisplit form of the unitary group \( PSU(2n) \) or \( PSU(2n+1) \), then \( \mathcal{G}(\tau) \) is the split form of \( SO(2n+1) \) or \( Sp(2n) \), respectively. In particular, we obtain a correspondence between the spherical unitary duals with real infinitesimal character of \( \mathcal{G} \) and \( \mathcal{G}(\tau) \). This identification of spherical unitary duals (by different methods) is also known to hold for the pairs of classical real groups \((U(n, n), SO(n+1, n))\) and \((U(n+1, n), Sp(2n, \mathbb{R}))\), see [Ba].

If \( \mathcal{G} \) does not split over an unramified extension of \( \mathbb{F} \), it is likely that one may apply the method used in [Lu7, section 10.13]. As it is shown there, for every such \( \mathcal{G} \), there exists a different group \( \mathcal{G}' \) which splits over an unramified extension of \( \mathbb{F} \), with the property that the Iwahori-Hecke algebra \( H(\mathcal{G}, 1_T) \) can be identified with the Iwahori-Hecke algebra \( H(\mathcal{G}', 1_T) \). Therefore the categories \( \mathcal{R}^1(\mathcal{G}) \) and \( \mathcal{R}^1(\mathcal{G}') \) are equivalent. (See [Lu7, page 280] for the list of pairs \((\mathcal{G}, \mathcal{G}')\).)

**1.5.** The second example is when \( 0^0 \chi \) is a nontrivial character of \( 0^0 M \). We rely on the theory of types results of [Ro] for ramified principal series, so we need to assume that \( \mathcal{G} \) is split, and have certain restrictions on the characteristic of \( \mathbb{F} \). In this case too, our methods imply a correspondence of endoscopic type (see section 6.3). To \( 0^0 \chi \) one attaches a semisimple element \( \hat{G}_X \) in the Langlands dual \( G \). Let \( C_G(\hat{G}_X) \) be the centralizer of \( \hat{G}_X \) in \( G \). This is a possibly disconnected reductive group. We define a dual split group \( \mathcal{G}'(0^0 \chi) \), the \( \mathbb{F} \)-points of a disconnected reductive group defined over \( \mathbb{F} \) (section 6.3).

**Theorem 1.5.1.** Let \( \mathcal{G} \) be a split group and \( 0^0 \chi \) a nontrivial character of \( 0^0 M \). In the isomorphism of categories \( \mathcal{R}^{0^0 \chi}(\mathcal{G}) \) and \( \mathcal{R}^1(\mathcal{G}'(0^0 \chi)) \) (from [Ro], see section 6.3), the hermitian and unitary representations correspond, respectively.
Acknowledgements. The authors thank M. Solleveld for pointing out several inaccuracies in a previous version of this paper. This research was supported in part by the NSF-DMS 0554278 and 0901104 for D.B., NSF-DMS 0968065 and NSA-AMS 081022 for D.C.

2. Affine Hecke algebras and graded affine Hecke algebras

In this section we recall the definitions of the affine Hecke algebra, its graded version, and the relation between their unitary duals. We follow [Lu1] and [BM2]. There are certain minor modifications because we need to consider extended Hecke algebras.

2.1. Let \( \Psi = (X, X^\vee, R, R^\vee) \) be a root datum. Thus \( X, X^\vee \) are two \( \mathbb{Z} \)-lattices with a perfect pairing \( \langle , \rangle : X \times X^\vee \rightarrow \mathbb{Z} \), the subsets \( R \subset X \setminus \{0\} \) and \( R^\vee \subset X^\vee \setminus \{0\} \) are in bijection \( \alpha \in R \leftrightarrow \alpha \in R^\vee \), satisfying \( \langle \alpha, \alpha \rangle = 2 \). For every \( \alpha \in R \), the reflections \( s_\alpha : X \rightarrow X, s_\alpha(x) = x - \langle x, \alpha \rangle \alpha \), and \( s_\beta : X^\vee \rightarrow X^\vee, s_\beta(y) = y - \langle \alpha, y \rangle \bar{\alpha} \), leave \( R \) and \( R^\vee \) stable respectively. Let \( W \) be the finite Weyl group, i.e. the group generated by the set \( \{s_\alpha : \alpha \in R\} \). We fix a choice of positive roots \( R^+ \), with basis \( \Pi \) of simple roots, and let \( R^\vee, \Pi \) be the corresponding images in \( R^\vee \) under \( \alpha \mapsto \bar{\alpha} \). We assume that \( \Psi \) is reduced (i.e., \( \alpha \in R \) implies \( 2\alpha \notin R \)). Let \( \ell \) be the length function of \( W \) with respect to the basis \( \Pi \) of simple roots.

The connected complex linear reductive group corresponding to \( \Psi \) is denoted \( G(\Psi) \) or just \( G \) if there is no danger of confusion. Then \( T := X^\vee \otimes_{\mathbb{Z}} \mathbb{C}^\times \) is a maximal torus in \( G \), and let \( B \supset T \) be the Borel subgroup such that the roots of \( T \) in \( B \) are \( R^+ \).

A parameter set for \( \Psi \) is a pair of functions \( (\lambda, \lambda^*) \),

\[
\lambda : \Pi \rightarrow \mathbb{Z}_{\geq 0}, \quad \lambda^* : \{\alpha \in \Pi : \bar{\alpha} \in 2X^\vee\} \rightarrow \mathbb{Z}_{\geq 0},
\]

such that \( \lambda(\alpha) = \lambda(\alpha') \) and \( \lambda^*(\alpha) = \lambda^*(\alpha') \) whenever \( \alpha, \alpha' \) are \( W \)-conjugate.

Definition 2.1.1. The affine Hecke algebra \( \mathcal{H}^{\lambda, \lambda^*}(\Psi, z) \), or just \( \mathcal{H}(\Psi) \), associated to the root datum \( \Psi \) with parameter set \( (\lambda, \lambda^*) \), is the associative algebra over \( \mathbb{C}[z, z^{-1}] \) with unit \( z \) is an indeterminate), defined by generators \( T_w, w \in W \), and \( \theta_x, x \in X \) with relations:

\[
(T_{s_\alpha} + 1)(T_{s_\alpha} - z^{2\lambda(\alpha)}) = 0, \quad \text{for all } \alpha \in \Pi, \tag{2.1.1}
\]

\[
T_wT_{w'} = T_{ww'}, \quad \text{for all } w, w' \in W \text{ such that } \ell(ww') = \ell(w) + \ell(w'),
\]

\[
\theta_x\theta_{x'} = \theta_{x + x'}, \quad \text{for all } x, x' \in X, \tag{2.1.2}
\]

\[
\theta_xT_{s_\alpha} - T_{s_\alpha}\theta_{s_\alpha(x)} = (\theta_x - \theta_{s_\alpha(x)})(G(\alpha) - 1), \quad \text{where } x \in X, \alpha \in \Pi, \text{ and}
\]

\[
G(\alpha) = \begin{cases} 
\frac{\theta_{z^{2\lambda(\alpha)-1}}}{\theta_{z^{\lambda(\alpha)+\lambda^*(\alpha)-1}}(\theta_{z^{2\lambda(\alpha)-1}})^{-1}}, & \text{if } \bar{\alpha} \notin 2X^\vee, \\
\frac{\theta_{z^{\lambda(\alpha)+\lambda^*(\alpha)-1}}}{\theta_{z^{2\lambda(\alpha)-1}}}, & \text{if } \bar{\alpha} \in 2X^\vee. 
\end{cases} \tag{2.1.3}
\]

Let \( \Gamma \) be a finite group endowed with a homomorphism \( \Gamma \rightarrow \text{Aut}(G, B, T) \), satisfying the property that \( \lambda(\gamma(\alpha)) = \lambda(\alpha) \) and \( \lambda^*(\gamma(\alpha)) = \lambda^*(\alpha) \), for all \( \gamma \in \Gamma \). Then we can form the extended affine Hecke algebra

\[
\mathcal{H}'(\Psi) := \mathcal{H}(\Psi) \rtimes \Gamma, \tag{2.1.4}
\]

by adding the generators \( \{T_{\gamma}\}_{\gamma \in \Gamma} \) and relations

\[
T_{\gamma}T_w = T_{\gamma(w)}T_{\gamma}, \quad T_{\gamma}T_{\gamma'} = T_{\gamma\gamma'}, \quad T_{\gamma}\theta_x = \theta_{\gamma(x)}T_{\gamma},
\]

for \( \gamma \in \Gamma, \ w \in W, \ x \in X. \)
Remarks 2.1.2. (1) The case $\Pi' \cap 2X^\vee \neq \emptyset$ can occur only if $R$ has a factor of type $B$.
(2) Since $\theta_x - \theta_{s_\alpha(x)} = \theta_x(1 - \theta_{s_\alpha})$, where $n = \langle x, \alpha \rangle$, the denominator of $G(\alpha)$ actually divides $\theta_x - \theta_{s_\alpha(x)}$.
(3) If $\lambda(\alpha) = c$ for all $\alpha \in \Pi$, and $\lambda^*(\alpha) = c$, for all $\alpha \in 2X^\vee$, we say that $H(\Psi)$ is a Hecke algebra with equal parameters. For example, assume that $c = 1$, and $z$ acts by $\sqrt{q}$. If $\Psi$ corresponds to $SL(2, \mathbb{C})$, the algebra is generated by $T := T_{s_\alpha}$ and $\theta := \theta_{\frac{1}{2}\alpha}$, where $\alpha$ is the unique positive root, subject to
\begin{equation}
(T + 1)(T - q) = 0; \tag{2.1.5}
\end{equation}
\begin{equation}
\theta T - T\theta^{-1} = (q - 1)\theta. \tag{2.1.6}
\end{equation}

On the other hand, if $\Psi$ corresponds to $PGL(2, \mathbb{C})$, then the generators are $T := T_{s_\alpha}$ and $\theta' := \theta_{\alpha}$ subject to
\begin{equation}
(T + 1)(T - q) = 0; \tag{2.2.1}
\end{equation}
\begin{equation}
\theta' T - T(\theta')^{-1} = (q - 1)(1 + \theta'). \tag{2.2.2}
\end{equation}

2.2. Recall $T = X^\vee \otimes_{\mathbb{C}} \mathbb{C}^\times$. Then
\[ X = \text{Hom}(T, \mathbb{C}^\times), \quad X^\vee = \text{Hom}(\mathbb{C}^\times, T). \]

Let $A = A(\Psi)$ be the algebra of regular functions on $\mathbb{C}^\times \times T$. It can be identified with the abelian $\mathbb{C}[z, z^{-1}]$-subalgebra of $H(\Psi)$ generated by $\{\theta_x : x \in X\}$, where for every $x \in X$, $\theta_x : T \to \mathbb{C}^\times$ is defined by
\begin{equation}
\theta_x(y \otimes \zeta) = \zeta^{(x,y)}, \quad y \in X^\vee, \quad \zeta \in \mathbb{C}^\times. \tag{2.2.3}
\end{equation}

If we denote by $H_W$ the $\mathbb{C}[z, z^{-1}]$-subalgebra generated by $\{T_w : w \in W\}$, then
\[ H(\Psi) = H_W \otimes_{\mathbb{C}[z, z^{-1}]} A \]
as a $\mathbb{C}[z, z^{-1}]$-module. An important fact is that as algebras,
\[ H_W \cong \mathbb{C}[W] \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}]. \]

In the case of an extended algebra, set $W' := W \times \Gamma$ and $H_{W'} := H_W \times \Gamma$. Then
\[ H'(\Psi) = H_{W'} \otimes_{\mathbb{C}[z, z^{-1}]} A \]
as a $\mathbb{C}[z, z^{-1}]$-module.

Theorem 2.2.1 (Bernstein-Lusztig, [Lu1, Proposition 3.11]). The center of $H(\Psi)$ is $Z = A^W$, i.e., the $W$-invariants in $A$. Similarly, the center of $H'(\Psi)$ is $A^{W'}$.

Let $\text{mod} H(\Psi)$ or $\text{mod} H'(\Psi)$ denote the categories of finite dimensional Hecke algebra modules. By Schur’s lemma, every irreducible module $(\pi, V)$ has a central character, i.e. there is a homomorphism $\chi : Z \to \mathbb{C}$ such that $\pi(z)v = \chi(z)v$ for every $v \in V$ and $z \in Z$. By Theorem 2.2.1, the central characters correspond to $W$-conjugacy (respectively $W'$-conjugacy) classes $(z_0, s) \in \mathbb{C}^\times \times T$. Then, we have:
\begin{equation}
\text{mod} H(\Psi) = \bigcup_{(z_0, s) \in \mathbb{C}^\times \times W \setminus T} \text{mod}_{(z_0, s)} H(\Psi), \tag{2.2.4}
\end{equation}
where $\text{mod}_{(z_0, s)} H(\Psi)$ is the subcategory of modules with central character (corresponding to) $(z_0, s)$. Let $\text{Irr}_{(z_0, s)} H(\Psi)$ be the set of isomorphism classes of simple objects in this category. One has the similar definitions for $H'(\Psi)$. Throughout the section, we will assume that $z_0$ is a fixed number in $\mathbb{R}_{>1}$. 
2.3. Fix a $W$–orbit $O$ of an element $\sigma \in T$, and denote by $O'$ the $W'$–orbit of $\sigma$. Then $O' = (O_1 = O) \sqcup O_2 \sqcup \cdots \sqcup O_m$ where each $O_i$ is a $W$–orbit. Following [Lu1, section 4] and [BM2, section 3], define the decreasing chain of ideals $I^k$, $k \geq 1$, in $A$ as

$$I := \{ f \text{ regular function on } \mathbb{C}^\times \times T : f(1, \sigma) = 0, \forall \sigma \in O \},$$

and $I^k$ the ideal of functions vanishing on $O$ to at least order $k$. Let $I^k := I^k \mathcal{H} = \mathcal{H}I^k$, $k \geq 1$, be the chain of ideals in $\mathcal{H}(\Psi)$, generated by the $I^k$'s. Similarly define $(I')^k$, and $(I')^k$ in $\mathcal{H}'(\Psi)$ using the orbit $O'$.

**Definition 2.3.1.** The affine graded Hecke algebra $\mathbb{H}_O(\Psi)$ is the associated graded object to the chain of ideals $\cdots \supset \tilde{I}^k \supset \cdots$ in $\mathcal{H}(\Psi)$. Similarly $\mathbb{H}_O(\Psi)$ is the graded object in $\mathcal{H}'(\Psi)$ for the filtration $\cdots \supset \tilde{(I')}^k \supset \cdots$.

Let $t = X^V \otimes \mathbb{C}$ be the Lie algebra of $T$, and let $t^* = X \otimes \mathbb{C}$ be the dual space. Extend the pairing $\langle \ , \ \rangle$ to $t^* \times t$. Let $A$ be the algebra of regular functions on $\mathbb{C} \oplus t$. Note that $A$ can be identified with $\mathbb{C}[r] \otimes_\mathbb{C} S(t^*)$, where $S(\ )$ denotes the symmetric algebra, and $r$ is an indeterminate. In the following $\delta$ denotes the delta function.

**Theorem 2.3.2 ([Lu1, BM2]).** The graded Hecke algebra $\mathbb{H}_O(\Psi)$ is a $\mathbb{C}[r]$–algebra generated by $\{ t_w : w \in W \}, S(t^*)$, and a set of orthogonal idempotents $\{ E_\sigma : \sigma \in O \}$ subject to the relations

$$t_w \cdot t_{w'} = t_{ww'}, \ w, w' \in W;$$

$$\sum_{\sigma \in O} E_\sigma = 1, \ E_{\sigma'}E_{\sigma''} = \delta_{\sigma',\sigma''}E_{\sigma'}, \ E_\sigma t_{s_\alpha} = t_{s_\alpha}E_{s_\alpha \sigma},$$

$$\omega \cdot t_{s_\alpha} - t_{s_\alpha} \cdot s_\alpha(\omega) = r g(\alpha)(\omega, \check{\alpha}), \ \text{where } \alpha \in \Pi, \omega \in t^*;$$

$$g(\alpha) = \sum_{\sigma \in O} E_\sigma \mu_\sigma(\alpha), \ \text{and}$$

$$\mu_\sigma(\alpha) = \begin{cases} 0, & \text{if } s_\alpha \sigma \neq \sigma, \\ 2\lambda(\alpha), & \text{if } s_\alpha \sigma = \sigma, \ \check{\alpha} \notin 2X^V, \\ \lambda(\alpha) + \lambda^*(\alpha)\theta_{-\alpha}(\sigma), & \text{if } s_\alpha \sigma = \sigma, \ \check{\alpha} \in 2X^V. \end{cases}$$

Notice that if $s_\alpha \sigma = \sigma$, then $\theta_{\alpha}(\sigma) = \theta_{-\alpha}(\sigma)$, or equivalently, $\theta_{\alpha}(\sigma) \in \{ \pm 1 \}$. This implies that for the parameters $\mu_\sigma(\alpha)$ we have $\mu_\sigma(\alpha) \in \{ 0, 2\lambda(\alpha), \lambda(\alpha) - \lambda^*(\alpha) \}$, for every root $\alpha \in \Pi$. In particular, in the case of equal parameters Hecke algebra, the only possibilities are $\mu_\sigma(\alpha) \in \{ 0, 2\lambda(\alpha) \}$.

**Remark 2.3.3.** An important special case is when $O$ is formed of a single ($W$–invariant) element $\sigma$. Then there is only one idempotent generator $E_\sigma = 1$, so it is suppressed from the notation. The algebra $\mathbb{H}_O(\Psi)$ is generated by $\{ t_w : w \in W \}$ and $S(t^*)$ subject to the commutation relation

$$\omega \cdot t_{s_\alpha} - t_{s_\alpha} \cdot s_\alpha(\omega) = r \mu_\sigma(\alpha)(\omega, \check{\alpha}), \ \alpha \in \Pi, \ \omega \in t^*,$$

where

$$\mu_\sigma(\alpha) = \begin{cases} 2\lambda(\alpha), & \text{if } \check{\alpha} \notin 2X^V, \\ \lambda(\alpha) + \lambda^*(\alpha)\theta_{-\alpha}(\sigma), & \text{if } \check{\alpha} \in 2X^V. \end{cases}$$

Still assuming that $\sigma$ is $W$–invariant, we have $\theta_{\alpha}(\sigma) \in \{ \pm 1 \}$, for all $\alpha \in \Pi$. If in fact $\sigma$ is in the center of group $G(\Psi)$, then $\theta_{\alpha}(\sigma) = 1$, for all $\alpha \in \Pi$. 

Let $\Psi$ be the root datum for Example 2.3.5. We will use the notation $\mathbb{H}_{\mu_\sigma}$ for the graded Hecke algebra in the particular case defined by equations (2.3.7) and (2.3.8).

**Example 2.3.5.** Let $\Psi$ be the root datum for $\text{PGL}(2, \mathbb{C})$, in the equal parameter case, and $\Gamma = \{1\}$. We present three cases:

1. $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. This is clearly $W$-invariant. Then $\mathbb{H}_{\mu_\sigma}$ is generated by $t = t_{s_\alpha}$ and $\omega$ subject to
   \[ t^2 = 1, \quad t\omega + \omega t = 2r(\lambda). \]  
   (2.3.9)

2. $\sigma = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. Since we are in $\text{PGL}(2, \mathbb{C})$, the element $\sigma$ is $W$-invariant. Note that $\theta_\alpha(\sigma) = -1$, and so $\mathbb{H}_{\mu_\sigma}$ is generated by $t$ and $\omega$ subject to
   \[ t^2 = 1, \quad t\omega + \omega t = 0. \]  
   (2.3.10)

3. $\sigma = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$, $\zeta^2 \neq \pm 1$. Then $W \cdot \sigma = \{\sigma, \sigma^{-1}\}$. The algebra $\mathbb{H}_\sigma$ is generated by $E_\sigma$, $E_{\sigma^{-1}}$, $t$, $\omega$ satisfying the following relations:
   \[ E_\sigma^2 = E_\sigma, \quad E_{\sigma^{-1}}^2 = E_{\sigma^{-1}}, \quad E_\sigma \cdot E_{\sigma^{-1}} = 0, \quad E_\sigma + E_{\sigma^{-1}} = 1, \]
   \[ t E_\sigma = E_{\sigma^{-1}} t, \quad t^2 = 1, \quad t\omega + \omega t = 0. \]  
   (2.3.11)

We can think of $\mathbb{H}_\sigma(\Psi)$ as the associative algebra defined by the relations (2.3.2)-(2.3.6). In particular if there is a homomorphism $\Gamma \rightarrow \text{Aut}(G, B, T)$, we can define
   \[ \mathbb{H}_\sigma'(\Psi) := \mathbb{H}_\sigma(\Psi) \rtimes \Gamma \]  
   (2.3.12)

as the associative algebra generated by $\{t_\gamma\}_{\gamma \in \Gamma}$ and the generators of $\mathbb{H}_\sigma$ satisfying (2.3.2)-(2.3.6), and in addition
   \[ t_\gamma t_w = t_{\gamma(w)} t_\gamma, \quad t_\gamma t_{\gamma'} = t_{\gamma \gamma'}, \quad \gamma, \gamma' \in \Gamma, \]
   \[ t_\gamma \omega = \gamma(\omega) t_\gamma, \quad w \in W, \omega \in t^*. \]  
   (2.3.13)

**Corollary 2.3.6.** There is a natural identification
   \[ \mathbb{H}_\sigma(\Psi) = \mathbb{H}_\sigma'(\Psi) = \mathbb{H}_\sigma(\Psi) \rtimes \Gamma, \]

where $\mathbb{H}_\sigma(\Psi), \mathbb{H}_\sigma'(\Psi)$ are as in Definition 2.3.1, and $\mathbb{H}_\sigma'(\Psi)$ is as in (2.3.12).

**Proof.** By (3) of Proposition 3.2 of [BM2], $\mathbb{H}_\sigma' = \bigoplus_{i=1}^n \mathbb{H}_{\gamma_i}$; recall that $\{O_i\}$ is the $W$-orbit partition of the $W'$-orbit $O'$. Each $T_\gamma$ induces (by grading) an algebra isomorphism
   \[ t_\gamma : \mathbb{H}_{O_i} \rightarrow \mathbb{H}_{\gamma O_i = O_j} \]  
   (2.3.14)

and therefore an automorphism
   \[ t_\gamma : \mathbb{H}_{\sigma'} = \bigoplus \mathbb{H}_{O_i} \rightarrow \bigoplus \mathbb{H}_{\gamma O_i = O_j} = \mathbb{H}_{\sigma'} \]  
   (2.3.15)

satisfying the required relations. We omit further details. \qed

Define
   \[ C_{W'}(\sigma) := \{ w \in W' : w\sigma = \sigma \}. \]  
   (2.3.16)

Then $W' \cdot \sigma = \{ w_j \cdot \sigma : 1 \leq j \leq n \}$, where $\{w_j = 1, w_2, \ldots, w_n\}$ are coset representatives for $W'/C_{W'}(\sigma)$. Then $\{E_\tau\} = \{E_{w_j\sigma} : 1 \leq j \leq n\}$, and from Theorem 2.3.7, $\mathbb{H}_{\sigma'} = \mathbb{C}[W'] \otimes (\mathcal{E}' \otimes \Lambda)$, as a $\mathbb{C}[\tau]$-vector space, where $\mathcal{E}'$ is the algebra generated by the $E_{w_j\sigma}$'s with $1 \leq j \leq n$. 

Proposition 2.3.7 ([Lu1, Proposition 4.5]). The center of $\mathbb{H}_O$ is $Z = (E \otimes \mathbb{A})^W$. The center of $\mathbb{H}'_O$ is $Z' = (E' \otimes \mathbb{A})^{W'}$.

It follows that the central characters of $\mathbb{H}'_O$ are parameterized by $C_{W'}(\sigma)$–orbits in $C \oplus t$. Similarly to the last paragraph in section 2.2, define the category $\text{mod}_{(r_0,x)}\mathbb{H}'_O$, where $r$ acts by $r_0 > 0$ and $x \in t$.

2.4. We describe the structure of $\mathbb{H}'_O(\Psi) = \mathbb{H}'_O(\Psi)$ in more detail. Fix a $\sigma \in O' \subset T$. We define a root datum $\Psi_\sigma = (X, R_\sigma, X^\vee, R^+_\sigma)$ with positive roots $R^+_\sigma$, defined as follows:

\[
R_\sigma = \begin{cases} 
\alpha \in R : \theta_\alpha(\sigma) = 1, & \text{if } \check{\alpha} \notin 2X^\vee, \\
\alpha \in R : \theta_\alpha(\sigma) = \pm 1, & \text{if } \check{\alpha} \in 2X^\vee 
\end{cases} \quad (2.4.1)
\]

\[
R^+_\sigma = R_\sigma \cap R^+, \quad (2.4.2)
\]

\[
R^+_\sigma = \{ \check{\alpha} \in R^\vee : \sigma \in R_\sigma \}. \quad (2.4.3)
\]

Note that $\Psi_\sigma$ is the root datum for $(C_G(\sigma)_0, T)$, with positive roots $R^+_\sigma$ with respect to the Borel subgroup $C_G(\sigma)_0 \cap B$. Define

\[
\Gamma_\sigma := \{ w \in C_{W'}(\sigma) : w(R^+_\sigma) = R^+_\sigma \}. \quad (2.4.4)
\]

There is a group homomorphism $\Gamma_\sigma \rightarrow \text{Aut}(C_G(\sigma)_0, C_G(\sigma)_0 \cap B, T)$ such that $\mu_\sigma(\gamma \alpha) = \mu_\sigma(\alpha)$, so the extended Hecke algebra

\[
\mathbb{H}'_{\mu_\sigma}(\Psi_\sigma) := \mathbb{H}_{\mu_\sigma}(\Psi_\sigma) \rtimes \Gamma_\sigma \quad (2.4.5)
\]

is well defined.

Recall the coset representatives $\{w_1, \ldots, w_n\}$ for $W'/C_{W'}(\sigma)$ from the paragraph after (2.3.16). Set

\[
E_{i,j} = t_{w_i^{-1}w_j}E_{w_j} = E_{w_j}t_{w_i^{-1}w_j}, \quad \text{for all } 1 \leq i, j \leq n, \quad (2.4.6)
\]

and let $M_n$ be the matrix algebra with basis $\{E_{i,j}\}$.

Theorem 2.4.1 ([Lu1]). There is a natural algebra isomorphism

\[
\mathbb{H}'_O(\Psi) \cong M_n \otimes_C \mathbb{H}'_{\mu_\sigma}(\Psi_{\sigma}) = M_n \otimes_C (\mathbb{H}_{\mu_\sigma}(\Psi_{\sigma}) \rtimes \Gamma_\sigma).
\]

Since the only irreducible representation of $M_n$ is the $n$–dimensional standard representation, one obtains immediately the equivalences of categories:

\[
\text{mod}_{(r_0,x)}\mathbb{H}'_O(\Psi) \cong \text{mod}_{(r_0,x)}\mathbb{H}'_{\mu_\sigma}(\Psi_\sigma). \quad (2.4.7)
\]

Remarks 2.4.2. (1) When $\Gamma = \{1\}$ and $X^\vee$ is generated by $R^\vee$, that is, when $\Psi$ is of simply connected type, or more generally, if $X^\vee$ is generated by $R^\vee \cup \frac{1}{2}R^\vee$ (which includes the case of factors of type $B$ as well), then $C_{W}(\sigma) \subset W_\sigma$, and so $\Gamma_\sigma = \{1\}$, for every $\sigma \in T$. In this case, $\mathbb{H}'_O(\Psi_\sigma) = \mathbb{H}_O(\Psi_\sigma)$, and there is no need to consider the extended graded Hecke algebras (2.4.5).

(2) When $\sigma$ is $W'$–invariant, then $n = 1$, and so $\mathbb{H}'_O(\Psi) \cong \mathbb{H}'_{\mu_\sigma}(\Psi_\sigma) = \mathbb{H}_{\mu_\sigma}(\Psi_\sigma) \rtimes \Gamma_\sigma$. 

2.5. In this section, we discuss the relation between $\mathcal{H}(\Psi)$ and $\mathbb{H}_O(\Psi)$. We need some definitions first.

The torus $T = X^\vee \otimes \mathbb{C}^\times$ admits a polar decomposition $T = T_e \times T_h$, where $T_e = X^\vee \otimes \mathbb{R}^1$, and $T_h = X^\vee \otimes \mathbb{R}_{>0}$. Consequently, every $s \in T$ decomposes uniquely into $s = s_e \cdot s_h$, with $s_e \in T_e$ and $s_h \in T_h$. We call an element $s_e \in T_e$ elliptic, and an element $s_h \in T_h$ hyperbolic. Similarly, $t = X^\vee \otimes \mathbb{C}$ admits the decomposition $t = t_{IR} \oplus t_R$ into an imaginary part $t_{IR} = X^\vee \otimes \mathbb{R}^1$ and a real part $t_R = X^\vee \otimes \mathbb{R}$.

We need to define certain completions of the Hecke algebras. The algebras $\mathbb{C}[r]$, $S$, and $\mathbb{C}[r] \otimes S$ consist of polynomial functions on $\mathbb{C}$, $t$ and $\mathcal{M} := \mathbb{C} \oplus t$, respectively. Let $\widehat{\mathbb{C}[r]}$, $\widehat{S}$, and $\widehat{\mathbb{C}[r]} \otimes \widehat{S}$ be the corresponding algebras of holomorphic functions. Let $\mathcal{K}$ and $\widehat{\mathcal{K}}$ be the fields of rational and meromorphic functions on $\mathcal{M}$. Finally set $\widehat{A} := A \otimes \widehat{S} \subset \widehat{\mathcal{K}}$, and

$$
\widehat{H}_O := \mathbb{C}[W] \otimes (\widehat{\mathbb{C}[r]} \otimes \widehat{A}),
$$

$$
\widehat{H}_O(\mathcal{K}) := \mathbb{C}[W] \otimes (\mathcal{E} \otimes \mathcal{K}) \supset \widehat{H}_O,
$$

$$
\widehat{H}_O(\widehat{\mathcal{K}}) := \mathbb{C}[W] \otimes (\mathcal{E} \otimes \widehat{\mathcal{K}}) \supset \widehat{H}_O(\mathcal{K}), \widehat{H}_O.
$$

We make the analogous definitions for $\mathbb{H}'$.

**Theorem 2.5.1** ([Lu1, section 5.2] and [BM2, Theorem 3.5]). The map

$$
\iota : \mathbb{C}[W] \otimes (\mathcal{E} \otimes \mathcal{K}) \longrightarrow \widehat{H}_O(\widehat{\mathcal{K}}),
$$

defined by

$$
\iota(E_\sigma) = E_\sigma, \quad \iota(f) = f, \quad f \in \widehat{\mathcal{K}},
$$

$$
\iota(t_\alpha) = (t_\alpha + 1)(\sum_{\sigma \in O} g_\sigma(\alpha)^{-1}E_\sigma) - 1,
$$

where

$$
g_\sigma(\alpha) = 1 + \mu_\sigma(\alpha)\alpha^{-1} \in \widehat{\mathcal{K}}, \quad (2.5.1)
$$

is an algebra isomorphism. Similarly for extended algebras, we have the analogous isomorphism $\iota' : \mathbb{C}[W'] \otimes (\mathcal{E} \otimes \widehat{\mathcal{K}}) \longrightarrow \widehat{H}_O(\widehat{\mathcal{K}})$, with $\iota'(\gamma) = \gamma$.

To every character $\chi$ of $\mathcal{Z}$ (the center of $\mathcal{H} = \mathcal{H}_W \otimes \mathcal{A}$), there corresponds a maximal ideal $\mathfrak{J}_\chi = \{z \in \mathcal{Z} : \chi(z) = 0\}$ of $\mathcal{Z}$. Define the quotients

$$
\mathfrak{A}_\chi := \mathfrak{A}/\mathfrak{A} \cdot \mathfrak{J}_\chi, \quad \mathcal{H}(\Psi)_\chi = \mathcal{H}(\Psi)/\mathcal{H}(\Psi) \cdot \mathfrak{J}_\chi. \quad (2.5.2)
$$

Similarly, consider the ideal $\mathfrak{I}_\chi$ for every character $\chi$ of $\mathcal{Z}$ in $\mathbb{H}_O$, and define the analogous quotients. Then

$$
\mathfrak{A}_\chi := \mathfrak{A}/(\mathfrak{A} : \mathfrak{I}_\chi) = \widehat{\mathfrak{A}}/(\widehat{\mathfrak{A}} : \widehat{\mathfrak{I}_\chi}) = \widehat{\mathfrak{A}}_\chi,
$$

$$
\mathbb{H}_\chi := \mathbb{H}_O/(\mathbb{H}_O : \mathfrak{I}_\chi) = \mathbb{H}_O/(\mathbb{H}_O : \mathfrak{I}_\chi) = \widehat{\mathbb{H}_\chi}. \quad (2.5.3)
$$

The similar definitions and formulas hold for $\mathcal{H}'$ and $\mathbb{H}'_O$.

The map

$$
\tau : \mathbb{C} \oplus t_{IR} \rightarrow C^\times \times T, \quad (r_0, \nu) \mapsto (z_0, s) = (e^{r_0}, \sigma \cdot e^\nu) \quad (2.5.4)
$$

is $C_W(\sigma)$–invariant. It matches the central characters

$$
\tau : \chi = C_W(\sigma) \cdot (r_0, \nu) \leftrightarrow \chi = W \cdot (z_0, s). \quad (2.5.5)
$$

Moreover, $\tau$ is a bijection onto the central characters of $\mathcal{H}$ with elliptic part in $O$. Similarly for extended algebras, we have a matching $\tau' : \chi' = C_W(\sigma) \cdot (r_0, \nu) \leftrightarrow \chi = W' \cdot (z_0, s)$ which is a bijection onto central characters of $\mathcal{H}'$ with elliptic part in $O'$. 
Proposition 2.5.2 ([BM2, Proposition 4.1]). The map \( \phi : A[z, z^{-1}] \to \hat{C}[r] \otimes \hat{A} \) defined by
\[
\phi(z) = e^r,
\phi(\theta_x) = \sum_{\sigma \in O} \theta_x(\sigma) E_\sigma e^x, \quad x \in X,
\]
is a \( C \)-algebra homomorphism which maps \( J_\chi \) to \( I_\chi \) and defines by passage to the quotients an isomorphism between \( A_\chi \) and \( K_\chi \).

The map \( \Phi : H \to \hat{H}_O(K) \) defined by
\[
\Phi(a) = \phi(a),
\Phi(T_\alpha + 1) = \sum_{\sigma \in O} E_\sigma(t_\alpha + 1)\phi(G_\alpha)g_\sigma(\alpha)^{-1}
\]
with \( G_\alpha \) as in Definition 2.1.1, and \( g_\sigma(\alpha) \) as in (2.5.1), induces an isomorphism between \( H_\chi \) and \( (\hat{H}_O)_{\chi} \).

The map \( \Phi \) depends on \( (r_0, \nu) \in M_\mathbb{R} := \mathbb{C} \oplus t_\mathbb{R} \). We write \( \Phi_{(r_0, \nu)} \) when we want to emphasize this dependence.

Theorem 2.5.3 ([BM2, Theorem 4.3]). Assume \( \sigma \in T_e \). Let \( \chi = W \cdot (e^{r_0} \cdot \nu) \), \( \chi = C_W(\sigma) \cdot (r_0, \nu) \) be as in (2.5.5), with \( (r_0, \nu) \in \mathbb{C} \oplus t_\mathbb{R} \). The isomorphism
\[
\Phi_{(r_0, \nu)} : H(\Psi)_\chi \cong \hat{H}_O(\Psi)_\chi
\]
from (2.5.3) is analytic in \( (r_0, \nu) \in \mathbb{C} \times t_\mathbb{R} \).

Both the proposition and the theorem hold with the obvious modifications for \( H' \) and \( \hat{H}' \).

2.6. The algebras \( H(\Psi) \) and \( \hat{H}_O(\Psi) \) have natural \(*\)-operations. These are complex conjugate involutive anti-automorphisms defined on generators as follows, [BM2, section 5].

For \( H(\Psi) \), the generators are \( z, T_w, w \in W \), and \( \theta_x, x \in X \) (Definition 2.1.1), and we set:
\[
z^* = z, \quad T^*_w = T_{w^{-1}}, \quad \theta^*_x = T_{w_0} \theta_{-w_0 x} T_{w_0}^{-1},
\]
where \( w_0 \) is the longest Weyl group element. In fact, \( H(\Psi) \) has a structure of normalized Hilbert algebra given by the inner product \( [x, y] = \epsilon(y^* x) \), where \( \epsilon : H(\Psi) \to \mathbb{C} \) is defined by \( \epsilon(T_w) = \delta_{w, 1} \). For \( H' \), \(*\) acts in addition by
\[
T^*_{\gamma} := T_{\gamma^{-1}},
\]
and set \( \epsilon(T_\gamma) = 0 \).

For \( \hat{H}_O(\Psi) \), recall that the generators are \( r, t_w, w \in W, \omega \in \mathfrak{t}^*, E_\sigma', \sigma' \in W \cdot \sigma \). The graded \(*\)-operation:
\[
r^* = r, \quad t^*_w = t_{w^{-1}}, \quad E'^*_\sigma = E_{\sigma'}^*,
\]
\[
\omega^* = -\omega + \sum_{\beta \in \mathfrak{t}^+} t_{\beta}(\omega, \beta) \sum_{\sigma' \in W \cdot \sigma} E_{\sigma' \mu_{\sigma'}(\beta)}, \quad \omega \in \mathfrak{t}^*_R,
\]
where
\[
if \ s = s_e \cdot s_h \in T = T_e \times T_h, \quad s^* := s_e \cdot s_h^{-1}.
\]
Following [BM2], we call $s$ such that $s^* ∈ W · s$ hermitian. Note that every elliptic element is hermitian, and therefore, under our assumption that $σ$ be elliptic, the $*$-operation on $H_σ(Ψ)$ is well-defined. For $H′$, define in addition

$$t^*_γ = t_{γ−1}. \quad (2.6.5)$$

Using these $*$-operations, we define hermitian and unitary modules for $H(Ψ)$ and $H_0(Ψ)$ as well as for the extended algebras $H′(Ψ)$ and $H′_0(Ψ)$.

**Proposition 2.6.1** ([BM2, Theorem 5.7]). In the setting of Theorem 2.5.3, the isomorphism $H(Ψ)_X ≅ H_0(Ψ)_X$ is compatible with the $*$-structures. In the equivalences of categories

$$\text{mod}_κ H(Ψ) ≅ \text{mod}_κ H_0(Ψ), \quad (2.6.6)$$

$$\text{mod}_κ H′(Ψ) ≅ \text{mod}_κ H′_0(Ψ), \quad (2.6.7)$$

the hermitian irreducible modules and the unitary irreducible modules correspond, respectively.

**Corollary 2.6.2.** The equivalence of categories

$$F : \text{mod}_κ H′_0(Ψ_σ) ≅ \text{mod}_κ H′(Ψ) \quad (2.6.8)$$

given by combining (2.6.6) and (2.4.7), takes hermitian irreducible modules to hermitian irreducible modules and unitary irreducible modules to unitary irreducible modules.

**Proof.** This is immediate from Proposition 2.5.2, since tensoring with the $n$-dimensional standard representation of $M_n$ preserves irreducible, hermitian, and unitary modules, respectively. \(\square\)

**Remark.** Corollary 2.6.1 effectively says that, in order to compute the unitary dual for $H′(Ψ)$, it is equivalent to compute the unitary dual with real central character for extended graded Hecke algebras $H′_0(Ψ_σ) = H_0(Ψ_σ) × Γ_σ$, for every conjugacy class of $σ ∈ T_s$ (elliptic semisimple elements). In the case when $σ$ is a central element of $G$, this means that the unitary dual in $\text{mod}_κ(Ψ_σ)$ with $s_e = σ$ is identified with the unitary dual with real central character for $H_1(Ψ_1)$.

For future purposes, we record here how the functor $F$ from (2.6.8) behaves with respect to the $W'$-structure. The first part is Corollary 3.4.(2) in [BM2], and the second part is immediate from the remark after Theorem 2.3.7.

**Corollary 2.6.3** ([BM2, Corollary 3.4]). As $\mathbb{C}[W′]$-modules:

$$F(\overline{V}) = \text{Ind}_{W′}^{W′′}(σ)(\overline{V}), \quad (2.6.9)$$

where $\overline{V} ∈ \text{mod}_κ H_′(Ψ_σ), F(\overline{V}) ∈ \text{mod}_κ H(Ψ)$.

In particular, if $σ$ is $W′$-invariant, then $F(\overline{V}) ≅ \overline{V}$ as $\mathbb{C}[W′]$-modules.

2.7. Let $V$ be a $H′(Ψ)$-module on which $z$ acts by $z_0 ∈ \mathbb{R}_{>1}$. For every $t ∈ T$, define a $A$-generalized eigenspace of $V$:

$$V_t = \{v ∈ V : \text{ for all } x ∈ X, (x(t) − θ_x)^k v = 0, \text{ for some } k ≥ 0\}. \quad (2.7.1)$$

We say that $t$ is a weight of $V$ if $V_t ≠ \{0\}$. Let $Φ(V)$ denote the set of weights of $V$. We have $V = \bigoplus_{t ∈ Φ(V)} V_t$ as $A$-modules.

**Definition 2.7.1.** We say that $V$ is tempered if, for all $x ∈ X^+ := \{x ∈ X : \langle x, α \rangle ≥ 0, \text{ for all } α ∈ R^+\}$, and all $t ∈ Φ(V)$, $|x(t)| ≤ 1$ holds.
Let $\mathcal{V}$ be a $H'_{\mu_s}(\Psi_{\sigma})$–module on which $r$ acts by $r_0 > 0$. For every $\nu \in t$, define a $\mathbb{A}$–generalized eigenspace of $\mathcal{V}$,

$$\mathcal{V}_\nu = \{ v \in \mathcal{V} : \text{for all } \omega \in S(t^*) , (\omega(\nu) - \omega)^k v = 0, \text{ for some } k \geq 0 \}.$$  \tag{2.7.2}

We say that $\nu$ is a weight of $\mathcal{V}$ if $\mathcal{V}_\nu \neq \{0\}$. Let $\Phi(\mathcal{V})$ denote the set of weights of $\mathcal{V}$. We have $\mathcal{V} = \oplus_{\nu \in \Phi(\mathcal{V})} \mathcal{V}_\nu$, as $\mathbb{A}$–modules.

**Definition 2.7.2.** We say that $\mathcal{V}$ is tempered if, for all $\omega \in \mathbb{A}^+ := \{ \omega \in S(t^*) : \langle \omega, \tilde{\alpha} \rangle \geq 0, \text{ for all } \alpha \in R^+ \}$, and all $\nu \in \Phi(\mathcal{V})$, we have $\langle \omega, \nu \rangle \leq 0$.

The two notions of tempered are naturally related, see [BM2, sections 6 and 7], also [Lu7, Lemmas 4.3 and 4.4].

**Lemma 2.7.3.** In the equivalence of categories from Corollary 2.6.3, the tempered modules correspond. Precisely, $\mathcal{V}$ is tempered if and only if $\mathcal{V} = \mathcal{F}(\mathcal{V})$ is tempered, where $\mathcal{F}$ is as in (2.6.8).

### 3. Geometric graded Hecke algebras and linear independence

**3.1.** In this section, we discuss the graded Hecke algebra $\mathbb{H} = \mathbb{H}_{\mu_s}(\Psi_{\sigma})$ defined by the relations in (2.3.7) and (2.3.8) for the case $\sigma = 1$ (and $\Gamma = \{1\}$). We recall its definition in this particular case. Let $(X, R, X^\vee, R^\vee)$ be a root datum for a reduced root system, with finite Weyl group $W$, and simple roots $\Pi$. The relations are in terms of the function $\mu$ in (2.3.8). The lattices $X$ and $X^\vee$ are not explicitly needed for the relations, just $t := X^\vee \otimes_{\mathbb{Z}} \mathbb{C}$ and $t^* = X \otimes_{\mathbb{Z}} \mathbb{C}$. Then $\mathbb{H} = \mathbb{H}^{\mu}(t^*, R)$ is generated over $\mathbb{C}[r]$ by $\{ t_w : w \in W \}$ and $\omega \in t^*$, subject to

$$t_w \cdot t_{w'} = t_{ww'}, \quad w, w' \in W; \tag{3.1.1}$$

$$\omega \cdot \omega' = \omega' \cdot \omega; \quad \omega, \omega' \in t^*,$$

$$\omega \cdot t_{s_{\alpha_i}} - t_{s_{\alpha_i}} \cdot s_{\alpha}(\omega) = 2r \mu_{\alpha}(\omega, \tilde{\alpha}), \quad \omega \in t^*, \alpha \in \Pi.$$

We will consider only Hecke algebras of geometric type, i.e., those arising by the construction of [Lu2]. We will recall this next, but let us record first what the explicit cases are. Assume the root system is simple. Then there are at most two $W$–conjugacy classes in $R^+$. Since $\mu$ is constant on $W$–conjugacy classes, it is determined by its values $\mu_s := \mu(\alpha_s)$ and $\mu_l := \mu(\alpha_l)$, where $\alpha_s$ is a short simple root, and $\alpha_l$ is a long simple root. For uniformity of notation in the table below, we say $\mu_s = \mu_l$ in the simply-laced case. Only the ratio of these parameters is important, and there are also obvious isomorphisms between types $B$ and $C$. The list is in Table 1.

**Table 1.** Lusztig’s geometric parameters

<table>
<thead>
<tr>
<th>Type</th>
<th>ratio $\mu_s/\mu_l$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_n$</td>
<td>1</td>
</tr>
<tr>
<td>$B_n$</td>
<td>$\mathbb{Z}<em>{&gt;0}, \mathbb{Z}</em>{&gt;0} + 1/2, \mathbb{Z}<em>{&gt;0} + 1/4, \mathbb{Z}</em>{&gt;0} + 3/4$</td>
</tr>
<tr>
<td>$D_n$</td>
<td>1</td>
</tr>
<tr>
<td>$E_{6,7,8}$</td>
<td>1</td>
</tr>
<tr>
<td>$F_4$</td>
<td>1, 2</td>
</tr>
<tr>
<td>$G_2$</td>
<td>1, 9</td>
</tr>
</tbody>
</table>
3.2. We review briefly the construction and classification of these algebras following the work of Lusztig. Let $G$ be a complex connected reductive group, with a fixed Borel subgroup $B$, and maximal torus $H \subset B$. The Lie algebras will be denoted by the corresponding German letters.

**Definition 3.2.1.** A cuspidal triple for $G$ is a triple $(L, C, \mathcal{L})$, where $L$ is a Levi subgroup of $G$, $C$ is a nilpotent adjoint $L$–orbit on the Lie algebra $\mathfrak{g}$, and $\mathcal{L}$ is an irreducible $G$–equivariant local system on $C$ which is cuspidal in the sense of [Lu5].

Let $\mathfrak{L}(G)$ denote the set of $G$–conjugacy classes of cuspidal triples for $G$. For example, $(H, 0, \text{triv}) \in \mathfrak{L}(G)$. Let us fix $(L, C, \mathcal{L}) \in \mathfrak{L}(G)$, such that $H \subset L$, and $P = LU \supset B$ is a parabolic subgroup. Let $T$ denote the identity component of the center of $L$. Set $W = N_G(L)/L$. This is a Coxeter group due to the particular form $L$ must have to allow a cuspidal local system.

**Definition 3.2.2** ([Lu2, section 2]). Let $\mathbb{H}(L, C, \mathcal{L}) := \mathbb{H}^\mu(\mathfrak{t}, R)$ define a graded Hecke algebra as in (3.1.1) where:

(i) $\mathfrak{t}$ is the Lie algebra of $T$;
(ii) $W = N_G(L)/L$;
(iii) $\mathfrak{r}$ is the reduced part of the root system given by the nonzero weights of $\text{ad}(\mathfrak{t})$ on $\mathfrak{g}$; it can be identified with the root system of the reductive part of $C_G(x)$, where $x \in C$;
(iv) $\mathfrak{r}^+$ is the subset of $\mathfrak{r}$ for which the corresponding weight space lives in $\mathfrak{u}$;
(v) the simple roots $\Pi = \{\alpha_i : i \in I\}$ correspond to the Levi subgroups $L_i$ containing $L$ maximally: $\alpha_i$ is the unique element in $\mathfrak{r}^+$ which is trivial on the center of $I_i$;
(vi) for every simple $\alpha_i$, $\mu_{\alpha_i} \geq 2$ is defined to be the smallest integer such that

$$\text{ad}(x)^{\mu_{\alpha_i} - 1} : l_i \cap u \rightarrow l_i \cap u \text{ is zero.}$$

(3.2.1)

Up to constant scaling of the parameter function $\mu$, all the algebras in table 1 appear in this way. The explicit classification of cuspidal triples when $G$ is simple, along with the corresponding values for the parameters $\mu_{\alpha_i}$ can be found in the tables of [Lu2, 2.13].

3.3. Consider the algebraic variety

$$\hat{\mathfrak{g}} = \{(x, g\mathfrak{P}) \in \mathfrak{g} \times G/P : \text{Ad}(g^{-1})x \in C + \mathfrak{t} + \mathfrak{u}\},$$

(3.3.1)
on which $G \times \mathbb{C}^\times$ acts via $(g_1, \lambda) : x \mapsto \lambda^{-2}\text{Ad}(g_1)x$, $x \in \mathfrak{g}$, and $g\mathfrak{P} \mapsto g_1g\mathfrak{P}$, $g \in G$.

If $\mathcal{V}$ is any $G \times \mathbb{C}^\times$–stable subvariety of $\hat{\mathfrak{g}}$, we denote by $\hat{\mathcal{V}}$ the preimage under the first projection $\text{pr}_1 : \hat{\mathfrak{g}} \rightarrow \mathfrak{g}$. Let $\mathfrak{g}_N$ denote the variety of nilpotent elements in $\mathfrak{g}$. We will use the notation

$$\mathcal{P}_e := \{e\} = \{g\mathfrak{P} : \text{Ad}(g^{-1})e \in C + \mathfrak{u}\},$$

for any $e \in \mathfrak{g}_N$. (The identification is via the second projection $\hat{\mathfrak{g}} \rightarrow G/P$.) Define also $\mathcal{P}_e^\circ = \{g\mathfrak{P} \in \mathcal{P}_e : \text{Ad}(g^{-1})s \in \mathfrak{p}\}$.

We consider the projections $\text{pr}_C : \hat{\mathfrak{g}} \rightarrow C$, $\text{pr}_C(x, g) = \text{pr}_C(\text{Ad}(g^{-1})x)$, and $\text{pr}_P : \hat{\mathfrak{g}} \rightarrow \mathfrak{g}$, $\text{pr}_P(x, g) = (x, g\mathfrak{P})$, where $\hat{\mathfrak{g}} = \{(x, g) \in \mathfrak{g} \times G : \text{Ad}(g^{-1})x \in C + \mathfrak{h} + \mathfrak{u}\}$. They are both $G \times \mathbb{C}^\times$–equivariant. Let $\mathcal{L}$ be the $G \times \mathbb{C}^\times$–equivariant local system on $\hat{\mathfrak{g}}$ defined by the condition $\text{pr}_C^\circ(\mathcal{L}) = \text{pr}_P^\circ(\mathcal{L})$, and let $\mathcal{L}^*$ be its dual local system.
The classification of simple modules for $\mathbb{H} = \mathbb{H}(L, C, \mathcal{L})$ is in [Lu2, Lu3, Lu4]. Let us fix a semisimple element $s \in \mathfrak{g}$ and $r_0 \in \mathbb{C}^\times$, and let $T = T_{s, r_0}$ be the smallest torus in $G \times \mathbb{C}^\times$ whose Lie algebra contains $(s, r_0)$. Let $\mathfrak{g}_{s, r_0}$ be the set of $T$-fixed vectors in $\mathfrak{g}$, namely

$$\mathfrak{g}_{s, r_0} = \{ x \in \mathfrak{g} : [s, x] = 2r_0 x \}.$$  

(3.3.2)

Let $C_G(s) \times \mathbb{C}^\times$ be the centralizer of $(s, r_0)$ in $G \times \mathbb{C}^\times$. Since $s \in \mathfrak{g}$, $C_G(s)$ is a Levi subgroup of $G$, hence this centralizer is connected.

The construction of standard modules is in equivariant homology ([Lu2, section 1]). For $(e, \psi)$, where $e \in \mathfrak{g}_{s, r_0}$, and $\psi \in \hat{A}(s, e)$, the standard geometric module is (see [Lu3, 10.7, 10.12]):

$$X_{(s, e, \psi)} = \text{Hom}_{A(s, e)}[\psi : H^1_s(\mathcal{P}_e, \hat{\mathcal{L}})] = \text{Hom}_{A(s, e)}[\psi : C_{(s, r_0)} \otimes_{H^1_{C_G(s)} G} H^1_{C_G(s) \times \mathbb{C}^\times} \mathcal{P}_e, \hat{\mathcal{L}}].$$

(3.3.3)

One considers the action of $A_G(e) = C_G(e)/C_G(e)^0$ on the homology $H^1_{e}(\mathcal{P}_e, \hat{\mathcal{L}})$, and let $\hat{A}(e)_\mathcal{L}$ denote the representations of $A(e)$ which appear in this way. Note that the natural map $A(s, e) \to A(e)$ is in fact an injection. Let $\hat{A}(s, e)_\mathcal{L}$ denote the representations of $A(s, e)$ which appear as restrictions of $\hat{A}(e)_\mathcal{L}$ to $A(s, e)$. By [Lu3, 8.17] and [Lu2, 8.10], $X_{(s, e, \psi)} \neq 0$ if and only if $\psi \in \hat{A}(s, e)_\mathcal{L}$. One can phrase the classification as follows.

**Theorem 3.3.1** ([Lu3, 8.10, 8.14, 8.17], [Lu4, 1.15]). A standard module $X_{(s, e, \psi)}$ is nonzero if and only if $\psi \in \hat{A}(s, e)_\mathcal{L}$. When $X_{(s, e, \psi)} \neq 0$, it has a unique irreducible quotient $L_{(s, e, \psi)}$. This induces a natural one-to-one correspondence

$$(\text{Irr}_{0} \mathbb{H}(L, C, \mathcal{L}) \leftrightarrow \{(s, e, \psi) : [s, e] = r_0 e, s \in \mathfrak{g} \text{ semisimple, } e \in \mathfrak{g}_N, \psi \in \hat{A}(s, e)_\mathcal{L}\}) / G.$$  

(3.3.4)

**3.4.** We are interested in the $W$-structure of standard modules. Let $e \in \mathfrak{g}_N$ be given. By [Lu5] (also [Lu8, section 24]), the homology group $H^1_e(\mathcal{P}_e, \hat{\mathcal{L}})$ carries a $A(e) \times W$ action. Moreover,

$$\mu(e, \phi) := \text{Hom}_{A(e)}[\phi, H^1_e(\mathcal{P}_e, \hat{\mathcal{L}})]$$

(3.4.1)

is an irreducible $W$-representation. The correspondence $\hat{A}(e)_\mathcal{L} \to \hat{W}, (e, \phi) \to \mu(e, \phi)$ is the generalized Springer correspondence of [Lu5], and it is a bijection. We summarize in the following statement the relevant results from [Lu5, Lu8] that we need.

**Theorem 3.4.1** ([Lu5, 6.5], [Lu8, 24.4]). Let $(L, C, \mathcal{L})$ be a cuspidal triple as before.

1. If $e \in \mathfrak{g}_N$, $\phi \in \hat{A}(e)_\mathcal{L}$, then $\mu(e, \phi)$ from (3.4.1) is an irreducible $W$-representation.
2. The $W$-representation $\mu(e, \phi)$ appears with multiplicity one in $H^1_e(\mathcal{P}_e, \hat{\mathcal{L}})$.
3. If the $W$-representation $\mu(e', \phi')$ occurs in $H^1_e(\mathcal{P}_e, \hat{\mathcal{L}})$, then $e' \in \overline{G \cdot e}$, and if $G \cdot e' = G \cdot e$, then necessarily $\phi' = \phi$.
4. If $\phi' \notin \hat{A}(e)_\mathcal{L}$, then $\text{Hom}_{A(e)}[\phi', H^1_e(\mathcal{P}_e, \hat{\mathcal{L}})] = 0$.

Claim (4) in this theorem is proved in [Lu8], as a consequence of a (generalized) Green polynomials algorithm.

To transfer these results to the $W$-structure of standard modules, via (3.3.3), we also need the deformation argument of [Lu3, 10.13].
Lemma 3.4.2 ([Lu3], 10.13). In the notation of (3.3.3), there is an isomorphism of $W$–representations

$$X_{(s,e,\psi)} = \text{Hom}_{A(s,e)}[\psi : H^1_1(\mathcal{P}_{e}, \hat{\mathcal{L}})] = \text{Hom}_{A(s,e)}[\psi : H^1_1(\mathcal{P}_{e}, \hat{\mathcal{L}})].$$

(3.4.2)

3.5. Now we can recall the classification of tempered modules for the geometric Hecke algebras as in [Lu4]. Fix $r_0 > 0$.

Definition 3.5.1. A semisimple element $\sigma \in g$ is called hyperbolic (resp. elliptic) if $\text{ad}(\sigma) : g \to g$ has only real (resp. imaginary) eigenvalues.

Theorem 3.5.2 ([Lu4, 1.21]). The module $L_{(s,e,\psi)}$, where $\psi \in \hat{A}(s,e)\mathcal{L}$, is a tempered module for $H(L, C, \mathcal{L})$ if and only if there exists a Lie triple $\{e, h, f\}$ in $g$, such that $[s, h] = 0$, $[s, f] = -2r_0f$, $[s, e] = 2r_0e$, and $s - r_0h$ is elliptic. In this case, $L_{(s,e,\psi)} = X_{(s,e,\psi)}$.

By $sl(2)$–theory, any middle element $h$ of a Lie triple is hyperbolic. The condition that $s - r_0h$ be elliptic, implies that if $s$ is hyperbolic, then in fact $s = r_0h$. In this case $A(e) = A(s,e)$. Recall also that there is a one-to-one correspondence between nilpotent $G$–orbits in $g$ and $G$–conjugacy classes of Lie triples. Finally, we may assume that $s$ is in $t$, and therefore the central character of the irreducible $\mathbb{H}(L, C, \mathcal{L})$–module $L_{(s,e,\psi)}$ is the ($W$–conjugacy class of the) projection on $t$ of $s$. (The notation is as in section 3.2.) Putting these together, we have the following corollary.

Corollary 3.5.3. The map

$$\{(e, \phi) : e \in g_N, \phi \in \hat{A}(e)\mathcal{L}\}/G \longleftrightarrow X_{(r_0h,e,\phi)},$$

(3.5.1)

is a one-to-one correspondence onto the set of isomorphism classes of tempered irreducible $\mathbb{H}(L, C, \mathcal{L})$–modules with real central character, on which $r$ acts by $r_0$.

We can now formulate the main result we will need in the rest of the paper. This is the generalization of the result in [BM1] for Hecke algebras with equal parameters, and it also appeared in an equivalent form in [Ci].

Proposition 3.5.4. Assume $r_0 \neq 0$. The lowest $W$–type correspondence

$$X_{(r_0h,e,\phi)} \to \mu(e, \phi)$$

(3.5.2)

between tempered modules in $\text{Irr}_{r_0}\mathbb{H}(L, C, \mathcal{L})$ with real central character and $\hat{W}$ is a bijection. Moreover, the map

$$X_{(r_0h,e,\phi)} \to X_{(r_0h,e,\phi)}|_{W}$$

(3.5.3)

is uni-triangular with respect to the closure ordering of nilpotent orbits and the lowest $W$–type map. In particular, the set of tempered modules with real central character in $\text{Irr}_{r_0}\mathbb{H}(L, C, \mathcal{L})$ are linearly independent in the Grothendieck group of $W$.

Proof. Using Lemma 3.4.2 and Theorem 3.4.1, we see that every tempered module $X_{(r_0h,e,\phi)}$ as in Corollary 3.5.3 has a unique lowest $W$–type (with respect to the closure ordering of nilpotent orbits) $\mu(e, \phi)$, and this appears with multiplicity one. The unitriangularity is also clear from the same results. \qed
3.6. We would like to extend Proposition 3.5.4 to graded affine Hecke algebras $\mathbb{H}' := \mathbb{H} \rtimes \Gamma$ of the type as in section 2.4, under the assumption that $\mathbb{H}$ is of geometric type.

We wish to study the $W'$-structure of tempered $\mathbb{H}'$-modules with real central character. For this, we need some elements of Clifford-Mackey theory, as in [RR]. We recall the general setting, for an extended algebra $K' := K \rtimes \Gamma$, which will then be specialized first to $K = \mathbb{C}[W]$ and $K' = \mathbb{C}[W']$, and then to $K = \mathbb{H}$ and $K = \mathbb{H}'$.

Let $K$ be a finite dimensional $\mathbb{C}$-algebra, with an action by algebra automorphisms by $\Gamma$, and set $K' = K \rtimes \Gamma$. If $V$ is a finite dimensional module of $K$, and $\gamma \in \Gamma$, then let $\gamma V$ denote the $K$-module with the action $x \cdot v := \gamma^{-1}(x)v$, $x \in K$, $v \in V$. Clearly, $V$ is irreducible if and only if $\gamma V$ is irreducible. Assume $V$ is an irreducible $K$-module. Define the inertia subgroup

$$\Gamma_V = \{ \gamma \in \Gamma : V \cong \gamma V \}. \quad (3.6.1)$$

Since $V$ is simple, the isomorphism $V \to \gamma V$, for $\gamma \in \Gamma_V$, is unique up to a scalar multiple. We fix a family of isomorphisms $\{ \tau_\gamma : V \to \gamma^{-1} V \}_{\gamma \in \Gamma_V}$, and define the factor set

$$\beta : \Gamma_V \times \Gamma_V \to \mathbb{C}^*, \text{ such that } \tau_\gamma \tau_{\gamma'} = \beta(\gamma, \gamma') \tau_{\gamma \gamma'}. \quad (3.6.2)$$

Let $(\mathbb{C}\Gamma_V)_{\beta^{-1}}$ be the algebra with basis $\{ \gamma : \gamma \in \Gamma_V \}$ and multiplication

$$\gamma \cdot \gamma' = \beta(\gamma, \gamma')^{-1}(\gamma \gamma'), \quad (3.6.3)$$

where the latter multiplication is understood in $\mathbb{C}\Gamma_V$. Up to algebra isomorphism, the algebra $(\mathbb{C}\Gamma_V)_{\beta^{-1}}$ is independent of the choice of the family $\{ \tau_\gamma \}$. If $U$ is any irreducible $(\mathbb{C}\Gamma_V)_{\beta^{-1}}$-module, then there is a natural $K \rtimes \Gamma_V$ action on $V \otimes U$: $x\gamma(v \otimes u) := (x\tau_\gamma v) \otimes (\gamma u)$, for $x \in K$, $\gamma \in \Gamma_V$.

**Theorem 3.6.1 ([RR, A.6]).** With the same notation as above, define the induced module

$$V \rtimes U := \text{Ind}_{K \rtimes \Gamma_V}^{K' \rtimes \Gamma_V} (V \otimes U).$$

Then

(a) $V \rtimes U$ is an irreducible $K \rtimes \Gamma-$module.

(b) Every irreducible $K \rtimes \Gamma-$module appears in this way.

(c) If $V \rtimes U \cong V' \rtimes U'$, then $V, V'$ are $\Gamma-$conjugate, and $U \cong U'$ as $(\mathbb{C}\Gamma_V)_{\beta^{-1}}-$modules.

We go back to our setting. Set $K = \mathbb{C}[W]$ first. For every $\mu \in \hat{W}$, let $\Gamma_\mu$ be the inertia group, and fix a family of isomorphisms $\{ a^\mu_\gamma : \mu \to \gamma^{-1} \mu \}_{\gamma \in \Gamma_\mu}$. Since the action of $\Gamma$ on $W$ comes from the action of $\Gamma$ on the root datum, we have the following lemma.

**Lemma 3.6.2.** In the notation of Theorem 3.4.1, if $\mu = \mu(e, \phi) \in \hat{W}$, for some $e \in \mathfrak{g}_N$ and $\phi \in \hat{\mathfrak{a}}(e)_{\mathcal{L}}$, then for every $\gamma \in \Gamma$ we have $\gamma \mu = \mu(\gamma e, \gamma \phi)$.

When $\mathbb{H}$ has equal parameters, this statement is a particular case of [Re, Propositions 2.6.1 and 2.7.3]. In more generality, one can follow the analogous argument using the construction of the $W$-action from [Lu2, 3.4, 8.1], and the formal functorial properties of equivariant homology from [Lu2, section 1].

Now, let us specialize to $K = \mathbb{H}$, so that $K' = \mathbb{H}'$. Assume $V$ is a tempered module of $\mathbb{H}$ with real central character. Then also $\gamma V$ is tempered with real central character for any $\gamma \in \Gamma$. By Proposition 3.5.4, there exists a unique lowest $W-$type of $V$, call it $\mu = \mu(e, \phi)$, which appears with multiplicity one. To emphasize this correspondence, we write $V = V(e, \phi)$. 
Proposition 3.6.3. We have $\Gamma_{V(e,\phi)} = \Gamma_{\mu(e,\phi)}$. Moreover, the factor set for $\Gamma_{V(e,\phi)}$ can be chosen to be equal to the factor set for $\Gamma_{\mu(e,\phi)}$.

Proof. Assume the $W-$structure of $V(e, \phi)$ is

$$V(e, \phi)|_W = \mu(e, \phi) \oplus \bigoplus_{e' \in \mathcal{G} \cdot e''} m(e',\phi') \mu(e',\phi').$$

(3.6.4)

Let $\gamma \in \Gamma$ be given. Then, by Lemma 3.6.2,

$$\mathcal{V}|_W = \mu(\gamma e, \gamma' \phi) \oplus \bigoplus_{e' \in \mathcal{G} \cdot e''} m(e',\phi') \mu(\gamma e',\gamma' \phi')$$

(3.6.5)

$$= \mu(\gamma e, \gamma' \phi) \oplus \bigoplus_{\gamma e' \in \mathcal{G} \cdot e''} m(e',\phi') \mu(e',\phi'),$$

where one uses the obvious fact that $e \in \mathcal{G} \cdot e''$ if and only if $\gamma e \in \mathcal{G} \cdot \gamma e''$. This means that the lowest $W-$type of $\mathcal{V}$ is $\gamma \mu$. By Proposition 3.5.4, it follows that $V \cong \gamma V$ if and only if $\mu \cong \gamma \mu$. This proves the first claim in the lemma.

For the second claim, let $\beta$ be the factor set for $\mu$ corresponding to the isomorphisms $\{a^\mu_\gamma\}_{\gamma \in \Gamma \mu}$. Let $\{\tau_\gamma : V \rightarrow \gamma^{-1} V\}_{\gamma \in \Gamma V}$ be a family of isomorphisms for $V$. Then by restriction to Hom-spaces, we get

$$\text{Hom}_W[ \mu : V ] \overset{\tau_\gamma}{\longrightarrow} \text{Hom}_W[ \gamma^{-1} \mu : \gamma^{-1} V ].$$  

(3.6.6)

By Theorem 3.4.1(2), these spaces are one-dimensional, and so the composition is a scalar. We normalize $\tau_\gamma$ so that this scalar equals to one. This forces $\{\tau_\gamma\}$ to have the same factor set $\beta$ as $\{a_\gamma\}$.

Corollary 3.6.4. There is a one-to-one correspondence $V' = V(e, \phi) \times U \rightarrow \mu'_{V'} = \mu(e, \phi) \times U$, $U \in \hat{\Gamma}_{\mu(e,\phi)}$, between tempered modules with real central character for $\mathbb{H}' = \mathbb{H} \rtimes \Gamma$ and representations of $W' = W \rtimes \Gamma$. Moreover, in the Grothendieck group of $W'$, the set of tempered $\mathbb{H}'-$modules with real central character is linearly independent.

Proof. The first claim follows immediately from Proposition 3.6.3, and also the fact that $\mu'_{V'}$ appears with multiplicity one in $V'$.

The second claim follows immediately, once we define a partial ordering on $\hat{W}'$ by setting $\mu'_{(e_1,\phi_1)} < \mu'_{(e_2,\phi_2)}$ if and only if $e_1 \in \mathcal{G} - e_2$. Then $\mu'_{V'}$ is the lowest $W'-type$ of $V'$ with respect to this order, and the restriction map $V' \rightarrow V'|_W$ is uni-triangular.

4. Signature characters

In the previous sections, we encountered three types of graded Hecke algebras associated to a root system, which appear naturally in the reductions of section 2.3 from the affine Hecke algebra attached to a root datum. There is the usual graded Hecke algebra $\mathbb{H}$, Definition (3.1.1), the extended Hecke algebra $\mathbb{H}' = \mathbb{H} \rtimes \Gamma$, by a group $\Gamma$ of automorphisms of the root system for $\mathbb{H}$, and also, an induced graded Hecke algebra, which we denote now $\tilde{\mathbb{H}}' = \mathcal{M}_n \otimes_{\mathbb{C}} \mathbb{H}'$, where $\mathcal{M}_n = \{E_{i,j}\}$ is a matrix algebra. The finite group parts of these algebras

\footnote{Recently, [So1] gave a proof, by homological methods, of the linear $W'$-independence of tempered modules with real central character, but not the "lowest $W'$-type" uni-triangularity, for graded affine Hecke algebras with arbitrary parameters.}
are denoted by $C[W], C[W'] = C[W \rtimes \Gamma]$, and $C[\tilde{W}'], \tilde{W}'$, respectively. Here $W'$ is a subgroup of index $n$ in $W$. In particular, the results of this section apply to section 2 by specializing $W$ to $W(\Psi_\sigma)$, the Weyl group of the root system in section 2.4. $\Gamma$ to $\Gamma_\sigma$ from (2.4.4), so that $W'$ is specialized to $C_{W'}(\sigma)$ from (2.3.16), and $\tilde{W}'$ to $W'$ from the paragraph before (2.2.3).

4.1. The Langlands classification for $\mathbb{H}$ is in [Ev, Theorem 2.1]. We need to formulate it in the setting of $\mathbb{H}' = \mathbb{H} \rtimes \Gamma$.

Recall that the graded Hecke algebra $\mathbb{H}$ corresponds to a root system $(t^*, R, t^*(R))$, simple roots $\Pi$, and parameter set $\mu$. Assume the indeterminate $r$ acts by $r_0 \neq 0$. Let $\Pi_P \subset \Pi$ be given, then we define a parabolic subalgebra $\mathbb{H}_P \subset \mathbb{H}$, which, as a $\mathbb{C}$-vector space is $\mathbb{H}_P = \mathbb{C}[W_P] \otimes S(t^*)$, where $W_P \subset W$ is the subgroup generated by the reflection in the roots of $\Pi_P$. The parameter set on $\mathbb{H}_P$ is obtained by restriction from $\mu$. Define

$$a_P = \{ x \in t : \langle \alpha, x \rangle = 0, \forall \alpha \in \Pi_P\}, \quad a_P^* = \{ \omega \in t^* : \langle \omega, \alpha \rangle = 0, \forall \alpha \in \Pi_P\},$$

$$t^*_M = \{ \omega \in t^* : \langle \omega, x \rangle = 0, \forall x \in a_P^*\}, \quad t_M = \{ x \in t : \langle \omega, x \rangle = 0, \forall \omega \in a_P\}. \quad (4.1.1)$$

Then $\mathbb{H}_P = \mathbb{H}_M \otimes S(a_P^*)$ as $\mathbb{C}$-algebras, where $\mathbb{H}_M = \mathbb{C}[W_P] \otimes S(t^*_M)$ as a vector space (and defining commutation relations coming from $\mathbb{H}$).

The Langlands classification for $\mathbb{H}$ takes the following form.

**Theorem 4.1.1** ([Ev, Theorem 2.1]).

(i) Let $V$ be an irreducible $\mathbb{H}$-module. Then $V$ is a quotient of a standard induced module $I(P,U,\nu) := \mathbb{H} \otimes_{\mathbb{H}_P} (U \otimes \mathbb{C}_\nu)$, where $U$ is a tempered $\mathbb{H}_M$-module, and $\nu \in a_P^*$, where

$$a_P^* = \{ x \in a_P : \langle \alpha, Re \nu \rangle > 0, \forall \alpha \in \Pi \setminus \Pi_P\}. \quad (4.1.2)$$

(ii) Any standard module $I(P,U,\nu)$ as in (i) has a unique irreducible quotient, denoted $J(P,U,\nu)$.

(iii) $J(P_1,U_1,\nu_1) \cong J(P_2,U_2,\nu_2)$ if and only if $\Pi_{P_1} = \Pi_{P_2}, U_1 \cong U_2$ as $\mathbb{H}_M$-modules, and $\nu_1 = \nu_2$.

Moreover, the quotient $J(P,U,\nu)$ is characterized by the following property. Define an order relation $>$ on real parameters $t_\mathbb{R}$ by

$$\nu > \nu_0 \text{ if } \langle \nu - \nu_0, \alpha \rangle > 0, \forall \alpha \in \Pi. \quad (4.1.3)$$

Then if $J(P_0,U_0,\nu_0)$ is an irreducible subquotient of $I(P,U,\nu)$ different than $J(P,U,\nu)$, we have $Rev > Rev_0$.

The case of $\mathbb{H}'$ compared to $\mathbb{H}$ is analogous to the case of a nonconnected $p$-adic group compared to its identity component. In the setting of $p$-adic groups, the Langlands classification for nonconnected groups with abelian group of components was carried out in [BJ1], starting with the known case of connected groups. We follow the same approach and simply translate into the graded Hecke algebra language the results from $p$-adic groups, the proofs being completely analogous.

To account for the action of $\Gamma$, define first

$$\Gamma_P = \{ \gamma \in \Gamma : \gamma \cdot \Pi_P = \Pi_P \}. \quad (4.1.4)$$

The problem is of course that while any two elements of $a_P^*$ are not conjugate under $W$, they may be conjugate under $\Gamma_P$. In order to address this, one can define an order $<$ on $\Pi$
and extend this as a lexicographic order with respect to \( \langle , \rangle \) on \( a_P \). Then we can consider the convex subchamber of \( a_P^+ \):

\[
a_P^+(\Gamma) = \{ x \in a_P^+ : x \leq \gamma \cdot x, \text{ for all } \gamma \in \Gamma_P \}. \tag{4.1.5}
\]

One can also define a variant of the order \( > \) from before ([BJ1, Lemma 3.3]). If \( \nu, \nu_0 \) are real parameters, then write \( \nu > C \nu_0 \) if there exist \( \gamma, \gamma_0 \in \Gamma \) such that \( \gamma \cdot \nu > \gamma_0 \cdot \nu_0 \), and one shows that this is well-defined.

**Definition 4.1.2.** We define a Langlands datum \( (P, U', \nu) \) for \( \mathbb{H}' \) to be a triple \( \nu, \mathbb{H}'_{P, \nu}, U', \nu \) where:

1. \( \Pi_P \subset \Pi \);
2. \( \nu \in a_P^+(\Gamma) \);
3. \( \mathbb{H}'_{P, \nu} = \mathbb{H}_P \rtimes \Gamma_{P, \nu} \), where \( \Gamma_{P, \nu} = \{ \gamma \in \Gamma_P : \gamma \cdot \nu = \nu \} \);
4. \( U' \) is irreducible tempered for \( \mathbb{H}'_{P, \nu} \).

Then the reformulation of the Langlands classification for non-connected \( p \)-adic groups in this setting is the following.

**Theorem 4.1.3** ([BJ1, Theorem 4.2], [BJ2, Theorem 3.4]). Assume that \( \Gamma \) is abelian.

(i) Let \( V' \) be an irreducible \( \mathbb{H}' \)-module. Then \( V' \) is a quotient of a standard induced module \( I(P, U', \nu) := \mathbb{H}' \otimes_{\mathbb{H}_{P, \nu}} (U' \otimes \mathbb{C}_\nu) \), for a Langlands datum \( (P, U', \nu) \) as in Definition 4.1.2.

(ii) Any standard module \( I(P, U', \nu) \) as in (i) has a unique irreducible quotient, denoted \( J(P, U', \nu) \). This appears with multiplicity one in \( I(P, U', \nu) \). Moreover, if \( (P_1, U'_1, \nu_1) \) is a Langlands datum for a different irreducible quotient than \( J(P, U', \nu) \), then \( \text{Rev} \nu > C \text{Rev} \nu_1 \).

(iii) \( J(P_1, U'_1, \nu_1) \cong J(P_2, U'_2, \nu_2) \) if and only if \( \Pi_{P_1} = \Pi_{P_2} \), \( \nu_1 = \nu_2 \), and \( U'_1 \cong U'_2 \) as \( \mathbb{H}'_{P, \nu} \)-modules.

**Remark 4.1.4.**

1. In the known applications to \( p \)-adic groups, the group \( \Gamma \) arises as the group of components of the centralizer of a semisimple element in a complex connected reductive group \( \mathcal{G} \). By a classical result of Steinberg ([St]), \( \Gamma \) can be embedded naturally into the fundamental group of \( \mathcal{G} \), and therefore it is abelian.

2. If \( \mathbb{H} \) is assumed simple, the only case not covered by Theorem 4.1.3 is when \( \mathbb{H} \) is of type \( D_4 \) (with equal parameters 1) and \( \Gamma = S_3 \). But in this case, the algebra \( \mathbb{H}' = \mathbb{H}(D_4) \times S_3 \) is known to be isomorphic with the graded Hecke algebra of type \( F_4 \) with parameters 1 on the long roots and 0 on the short roots. Therefore, with this identification, one may use Theorem 4.1.1 directly in that case. There is another instance of this phenomenon: if \( \mathbb{H}' = \mathbb{H}(A_1)^n \times S_n \), where \( S_n \) acts by permuting the \( A_1 \)-factors, then \( \mathbb{H}' \) is isomorphic with the graded Hecke algebra of type \( C_n \) with parameters 0 on the short roots, and 1 on the long roots.

3. In a recent preprint of Solleveld, [So2, Lemma 2.2.6(b)], a slightly different form of the maximality of the Langlands parameter from Theorem 4.1.3(ii) is proved for arbitrary \( \Gamma \), namely that \( ||\nu|| > ||\nu_1|| \).

One can transfer the classification to \( \mathbb{H}' \) as well, via the functor

\[
\mathcal{T} : (\mathbb{H}' - \text{mod}) \to (\mathbb{H}' - \text{mod}), \quad V' \mapsto \tilde{V}' := \mathbb{C}^n \otimes_{\mathbb{C}} V', \tag{4.1.6}
\]

where \( \mathbb{C}^n = \text{span}_\mathbb{C} \{ E_{i,1} : i = 1, n \} \) is, up to isomorphism, the unique irreducible module of \( \mathcal{M}_n \).
4.2. Let $\mathcal{R}$ denote any one of the three algebras and let $\mathcal{W}$ denote its finite part. One of the arguments below relies on an induction on the length of the parameter in Langlands classification. When $\Gamma$ is abelian or if $\mathbb{H}$ is simple, we can use Theorem 4.1.3(ii) and Remark 4.1.4(2). In more generality, one can use the maximality of the Langlands parameter in the form from Remark 4.1.4(3). For every irreducible $\mathcal{R}$–module, we define the notions of $\mathcal{W}$–character and signature. The idea is due to [Vo], and it was used in the $p$–adic and Hecke algebra cases by [BM1].

Definition 4.2.1. Let $(\pi, V)$ be a finite dimensional $\mathcal{R}$–module. For every irreducible representation $(\delta, V_\delta)$ of $\mathcal{W}$, set $V(\delta) = \text{Hom}_\mathcal{W}[V_\delta, V]$, and let $m(\delta) = \dim \mathbb{C}V(\delta)$ be the multiplicity of $\delta$ in $V$. We define the $\mathcal{W}$–character of $(\pi, V)$ to be the formal combination

$$\theta_\mathcal{W}(V) = \sum_{\delta \in \hat{\mathcal{W}}} m(\delta)\delta. \quad (4.2.1)$$

It is clear that $(\pi, V, \langle , \rangle)$ is unitary if and only if $q(\delta) = 0$ for all $\delta \in \hat{\mathcal{W}}$. The fundamental result that one needs relates the signature of an irreducible module to a combination of signatures of the tempered modules. In the original real groups setting, this is [Vo, Theorem 1.5]. In the setting of representations of $p$–adic groups with Iwahori fixed vectors and affine Hecke algebras, this is [BM1, Theorem 5.2].

Theorem 4.2.3 ([Vo, BM1]). Let $(\pi, V)$ be an irreducible $\mathcal{R}$–module having a nonzero hermitian form. Then there exist finitely many irreducible tempered modules $(\pi_j, V_j)$, and integers $a_j^\pm, j = 1, m$, such that

$$\Sigma(V) = \left( \sum_{\delta \in \hat{\mathcal{W}}} p(\delta)\delta, \sum_{\delta \in \hat{\mathcal{W}}} q(\delta)\delta \right). \quad (4.2.2)$$

4.3. We compare the signature characters for the Hecke algebras $\mathcal{R}$. Let $\chi$ denote a central character.

Definition 4.3.1. Let $(\pi, V)$ be an irreducible $\mathcal{R}$–module with a nonzero hermitian form and corresponding signature character $\Sigma(V)$ as in Theorem 4.2.3. We say that $V$ has the
Vogan property, if
\[ \sum_{j=1}^{m} a_j^\sigma \theta_\mathcal{W}(V_j) = 0 \text{ implies that } a_j^\sigma = 0, \text{ for all } j = 1, m. \]

We say that a subcategory \( \text{mod}_\varpi \mathcal{R} \) has the Vogan property if every irreducible module in this subcategory does.

**Lemma 4.3.2.** Let \( \mathcal{R} \) be \( \mathbb{H} \) or \( \mathbb{H}' \). Then the category of finite dimensional \( \mathcal{R} \)-modules with real central character has the Vogan property.

**Proof.** This is immediate by Proposition 3.5.4 and Corollary 3.6.4, which say that the set of tempered \( \mathcal{R} \)-modules with real central character have linearly independent \( \mathcal{W} \)-characters. \( \Box \)

We cannot apply the same argument for \( \tilde{\mathbb{H}}' \). The reason is the following. Note that, by Corollary 2.6.3, in the correspondence (4.1.6), we have
\[ \theta_{\mathcal{W}'}(\tilde{V}') = \text{Ind}_{\mathcal{W}}(\theta_{\mathcal{W}}(V')). \] (4.3.1)

Consequently, it is possible to have distinct tempered modules with real central character for \( \tilde{\mathbb{H}}' \), which have the same \( \mathcal{W}' \)-structure. In particular, the linear \( \mathcal{W}' \)-independence for tempered modules does not hold anymore.

**Example 4.3.3.** In the notation of section 2, let \( \mathcal{H}(\Psi, \sqrt{q}) \) be the affine Hecke algebra for the root datum \( \Psi \) of \( G = \text{Sp}(2m, \mathbb{C}) \) and constant parameter functions \( \lambda = \lambda^* \equiv 1 \). Set \( \Gamma = \{ 1 \} \). Let \( \sigma \) be an elliptic semisimple element of \( G \) such that \( C_G(\sigma) = \text{Sp}(2k, \mathbb{C}) \times \text{Sp}(2m-2k, \mathbb{C}) \). We have \( \Gamma_\sigma = \{ 1 \} \), \( \Psi_\sigma \) is the root datum of \( C_G(\sigma) \), and \( \mu_\sigma \) is the constant function 2. In the notation of this section: \( \hat{\mathcal{W}}' = W(C_m) \) and \( \mathcal{W}' = W(C_k) \times W(C_{m-k}) \), \( \mathbb{H}' = \mathbb{H}_\mu(\Psi_\sigma) = \mathbb{H}(C_k) \times \mathbb{H}(C_{m-k}) \), and \( \hat{\mathbb{H}}' \cong M_n \otimes \mathbb{C} \mathbb{H}' \), where \( n = \binom{m}{k} \). By Proposition 3.5.4, there are \(|\hat{W}(C_k)||\hat{W}(C_{m-k})|\) distinct tempered modules with real central character for \( \hat{\mathbb{H}}' \), and by Lemma 2.7.3, these give rise precisely to the tempered modules with real central character for \( \mathbb{H}' \). Therefore, whenever we have \(|\hat{W}(C_k)||\hat{W}(C_{m-k})| > |\hat{W}(C_n)|\), the tempered \( \mathbb{H}' \)-modules with real central character are not linearly \( \mathcal{W}' \)-independent. For example, this is the case if \( k = 2 \) and \( m = 4 \). One bypasses this difficulty as in [BM2].

**Proposition 4.3.4.** The category of finite dimensional \( \hat{\mathbb{H}}' \)-modules with real central character has the Vogan property.

**Proof.** Let \( \tilde{V}' \) be an irreducible hermitian \( \hat{\mathbb{H}}' \)-module with real central character. Assume that
\[ \Sigma(\tilde{V}') = (\sum_{j=1}^{m} a_j^+ \theta_{\mathcal{W}}(\tilde{V}_j'), \sum_{j=1}^{m} a_j^- \theta_{\mathcal{W}}(\tilde{V}_j')). \]

Applying the comparison theorems for parabolic induction and tempered spectra from [BM2, section 7, Theorem 7.2 and Corollary 7.3] to the functor \( V' \mapsto \tilde{V}' \) in (4.1.6), one finds that
\[ \Sigma(V') = (\sum_{j=1}^{m} a_j^+ \theta_{\mathcal{W}}(V_j'), \sum_{j=1}^{m} a_j^- \theta_{\mathcal{W}}(V_j')). \]
By Lemma 2.7.3, all $V'_j$ are tempered with real central character. If $\sum_{j=1}^{m} a_j \theta_{\tilde{W}'_j} (V'_j) = 0$, then $\tilde{V}'$ unitary, and so $V'$ is unitary too. This means $\sum_{j=1}^{m} a_j \theta_{W}(V'_j) = 0$. The claim follows now by Lemma 4.3.2.

4.4. The main application is next. Call an affine algebra $H(\Psi, z_0)$ of geometric type, if all the graded algebras $H_{\mu, \alpha}$ that appear via the reductions in section 2 are of geometric type in the sense of section 3.

**Theorem 4.4.1.** Assume the notation from section 2. Let $H(\Psi, z_0)$ be the affine Hecke algebra attached to a root datum $\Psi$ (Definition 2.1.1) and with parameters $\lambda, \lambda^*$ of geometric type (section 3), and assume $z_0$ is not a root of unity, and let $H(\Psi, z_0) = H(\lambda, \Psi, z_0) \rtimes \Gamma$ be an extended Hecke algebra. Let $G$ be a fixed elliptic semisimple element. The category of finite dimensional $H(\lambda, \Psi, z_0)\rtimes \Gamma$-modules whose central characters have elliptic parts $G(\Psi) \rtimes \Gamma$-conjugate to $s_e$ has the Vogan property (Definition 4.3.1).

**Proof.** This follows now from Propositions 2.5.2, 2.6.1, and 4.3.4. □

5. Elements of the theory of types

We need to recall certain elements of the theory of types. The background material for section 5 is well-known, e.g., [BK1, BK2, BHK].

5.1. Let $F$ denote a $p$-adic field, with norm $|||$, and let $G$ be the group of $F$-rational points of a connected, reductive, algebraic group defined over $F$. Let $\mathfrak{R}(G)$ denote the category of smooth complex representations of $G$.

A character $\chi : G \to \mathbb{C}^\times$ is called unramified, if there exist $F$-rational characters $\phi_j : G \to \mathbb{F}_p^\times$, $j = 1, k$, and complex numbers $s_j, j = 1, k$, such that $\chi(g) = \prod_{j=1}^{k} ||\phi_j(g)||^{s_j}$, for all $g \in G$.

We recall Bernstein’s decomposition of $\mathfrak{R}(G)$ adapted to the theory of types. One defines an equivalence relation on the set of pairs $(L, \sigma)$, where $L$ is a $F$-rational Levi subgroup of $G$, and $\sigma$ is an irreducible supercuspidal representation of $L$ as follows.

**Definition 5.1.1.** Two pairs $(L_1, \sigma_1)$ and $(L_2, \sigma_2)$ are inertially equivalent if there exists $g \in G$, and an unramified character $\chi$ of $L_2$ such that $gL_1g^{-1} = L_2$, and $g \cdot \sigma_1 = \sigma_2 \circ \chi$.

Let $\mathfrak{B}(G)$ denote the set of inertial equivalence classes. If $(\pi, V)$ is an irreducible representation in $\mathfrak{R}(G)$, then there exists a pair $(P, \sigma)$, where $P = LN$ is a $F$-rational parabolic and $\sigma$ is an irreducible supercuspidal representation of $L$, such that $\pi$ is equivalent with a subquotient of the normalized induced representation $\text{Ind}_P^G(\sigma)$. Moreover, the pair $(L, \sigma)$ is unique up to conjugacy, so in particular, one can assign to $\pi$, in a well-defined way, the inertial class $s = [L, \sigma] \in \mathfrak{B}(G)$. This is called the inertial support of $\pi$. More generally, one defines a full subcategory $\mathfrak{R}^s(G)$ whose objects are $(\pi, V)$ (not necessarily irreducible) for which all irreducible subquotients have inertial support $s$.

**Remarks 5.1.2.** (1) The inertial support gives a decomposition

$$\mathfrak{R}(G) = \prod_{s \in \mathfrak{B}(G)} \mathfrak{R}^s(G).$$

(5.1.1)
(2) This decomposition is well behaved with respect to Langlands classification. Since all subquotients of a parabolically induced module have the same inertial support ([BZ, Theorem 2.9]), if the Langlands subquotient of a standard module has inertial support $s$, then the standard module is in $\mathcal{R}^s(G)$.

5.2. Let $\mathcal{H}(G)$ denote the Hecke algebra of $G$, i.e. the space of locally constant, compactly supported complex functions on $G$ with convolution with respect to some fixed Haar measure.

The algebra $\mathcal{H}(G)$ has a natural $*$-operation: if $f \in \mathcal{H}(G)$, then $f^*(g) := \overline{f(g^{-1})}$. Let $J$ be a compact open subgroup of $G$, and fix a smooth irreducible representation $(\rho, W)$ of $J$. Let $(\rho^\vee, W^\vee)$ be the contragredient representation.

**Definition 5.2.1.** Let $\mathcal{H}(G, \rho)$ be the vector space of compactly supported functions $f : G \to \text{End}_C(W^\vee)$ satisfying

$$f(j_1gj_2) = \rho^\vee(j_1) \circ f(g) \circ \rho(j_2), \quad j_1, j_2 \in J, g \in G.$$ 

Under the convolution, $\mathcal{H}(G, \rho)$ becomes an associative algebra with unit given by the function

$$1_\rho(g) = \frac{1}{\text{vol}(J)} \rho^\vee(g), \quad \text{if } g \in J, \quad \text{and } 1_\rho(g) = 0, \quad \text{if } g \notin J.$$ 

Let $^*$ denote the adjoint involution on $\text{End}_C(W^\vee)$ with respect to a fixed $J$-invariant positive definite form on $W^\vee$. Then one can define $*$-operation on $\mathcal{H}(G, \rho)$ by

$$f^*(g) := f(g^{-1})^*, \quad f \in \mathcal{H}(G, \rho), \quad g \in G,$n

and an inner product $[ , ]$ given by

$$[f, h] := \frac{\text{vol}(J)}{\dim \rho} \text{tr}((f^* \star h)(1_G)), \quad f, h \in \mathcal{H}(G, \rho);$$

here $1_G$ denote the identity in $G$. This makes $\mathcal{H}(G, \rho)$ into a normalized Hilbert algebra with an abstract Plancherel formula, see [BHK, sections 3 and 4]. In particular, one can define tempered, hermitian and unitary $\mathcal{H}(G, \rho)$-modules.

There is a natural isomorphism $\mathcal{H}(G, \rho) \cong \text{End}_G(c\text{-Ind}_J^G(\rho))$, where $c\text{-Ind}$ denotes compact induction. If $(\pi, V) \in \mathcal{R}(G)$, define the space of $\rho$-invariants in $V$:

$$V_\rho = \text{Hom}_J[W, V].$$

This space is a left $\mathcal{H}(G, \rho)$-module as follows. If $\phi \in \text{Hom}_J[W, V]$ and $f \in \mathcal{H}(G, \rho)$, then $\pi(f)\phi$ is the homomorphism

$$W \ni w \mapsto \int_G \pi(g) \phi((f(g)^\vee w) \ dg.$$ 

5.3. Let $\mathcal{R}_\rho(G)$ denote the full subcategory of $\mathcal{R}(G)$ whose objects are representations $(\pi, V)$ such that $V$ is generated by the $\rho$-isotypic component of $V$. One has the functor:

$$\mathcal{M}_\rho : \mathcal{R}_\rho(G) \to \mathcal{H}(G, \rho) \mod,$n

$$\mathcal{M}_\rho(V) = V_\rho.$$ 

**Definition 5.3.1.** The pair $(J, \rho)$ is called a type in $G$ if the category $\mathcal{R}_\rho(G)$ is closed under subquotients.

**Theorem 5.3.2** ([BK2, 3.12,4.3]). Assume that $(J, \rho)$ is a type in $G$. Then:

(i) the functor $\mathcal{M}_\rho$ from (5.3.1) is an equivalence of categories;

(ii) there exists a finite subset $\mathcal{S} \subset \mathcal{B}(G)$ such that $\mathcal{R}_\rho(G) = \prod_{s \in \mathcal{S}} \mathcal{R}^s(G)$.

(In this case, $(J, \rho)$ is called an $\mathcal{S}$-type.)

It is known from [BHK] that the functor $\mathcal{M}_\rho$ induces a homeomorphism of the supports of the Plancherel measures in the two categories. In particular:
Theorem 5.3.3 ([BHK, Theorem B]). The functor $M_\rho$ induces a bijection between the irreducible tempered modules in $R(\rho(G))$ and $H(G, \rho) - \text{mod}$.

5.4. Let $(J, \rho)$ be a $s$-type, where $s = [L, \sigma] \in B(\mathcal{G})$. The pair $(L, \sigma)$ also gives rise to an element $s_L = [L, \sigma] \in B(L)$. Assume that there exists a $s_L$-type $(J_L, \rho_L)$ in $L$ such that $(J, \rho)$ is a cover of $(J_L, \rho_L)$, in the sense of [BK2, section 8]. Let $P$ be a parabolic subgroup of $G$ with Levi $L$. By [BK2, (7.12)], there exists a natural injective algebra homomorphism

$$t_P : H(L, \rho_L) \to H(G, \rho). \quad (5.4.1)$$

If the supercuspidal $L$-representation $\sigma$ is obtained by compact induction as in [BK2, (5.5)], then the algebra $H(L, \rho_L)$ is abelian, [BK2, Proposition 5.6].

Definition 5.4.1. The $s$-type $(J, \rho)$ is called affine if there exists an algebra isomorphism

$$\xi : H(G, \rho) \to H'(\Psi),$$

where $H'(\Psi)$ is an extended affine Hecke algebra from Definition 2.1.1 of geometric type satisfying the properties:

(i) $\xi$ is an isomorphism of Hilbert algebras, where the Hilbert algebra structure for $H(G, \rho)$ is as in Definition 5.2.1, while for $H'(\Psi)$ it is as in section 2.6.

(ii) there exists a compact open subgroup $K$ of $G$ with $J \subset K$ and $G = KP$, such that $\xi(H(K, \rho)) = H_W$, where $H_W$ is as in (2.2.2).

(iii) $\xi(t_P(H(L, \rho_L))) = A(\Psi)$, where $A(\Psi)$ is as in section 2.2.

The following remark justifies condition (i) in Definition 5.4.1.

Remark 5.4.2. If $(J, \rho)$ satisfies condition (i) in Definition 5.4.1, then the isomorphism $\xi$ induces a bijection between the irreducible tempered $H(G, \rho)$-modules and the irreducible tempered $H'(\Psi)$-modules.

Proof. The tempered spectrum of $H(G, \rho)$ is the support of the Plancherel measure in $H(G, \rho)$-mod. Furthermore, the tempered spectrum for $H'(\Psi)$ defined in Definition 2.7.1 is also the support of the Plancherel measure for $H'(\Psi)$ (see [Op, Lemma 2.20 and Theorem 2.25]). The claim now follows from property (i) of $\xi$ in Definition 5.4.1. \qed

5.5.

Definition 5.5.1. Two module categories (as before) are said to be unitarily equivalent if their unitary, respectively hermitian irreducible modules are in bijection via an equivalence of categories.

We can extend [BM1, Theorem 1.1] to obtain the correspondence of hermitian and unitary irreducible modules between the categories $R_\rho(G) = R^\mathfrak{g}(G)$ and $H'(\Psi) - \text{mod} = H(G, \rho) - \text{mod}$. With Theorem 4.4.1, Theorem 5.3.3 and Remark 5.4.2 in hand, the argument of [BM1, Theorem 1.1] from the Iwahori case for split adjoint groups can be applied formally to the setting of an affine type $(J, \rho)$.

Theorem 5.5.2. If the type $(J, \rho)$ is affine in the sense of Definition 5.4.1, then the categories $R^\mathfrak{g}(G) = R_\rho(G)$ and $H(G, \rho) - \text{mod} = H'(\Psi) - \text{mod}$ are unitarily equivalent via the functor $M_\rho$.

\[ \text{for some specialized value } z_0 \in \mathbb{R}_{>1} \text{ of the indeterminate } z. \]
Sketch of proof. The nontrivial part is to prove that the unitarity of the $\mathcal{H}(G, \rho)$-module implies that unitarity of the representation in $\mathbb{R}^f(G)$. Definition 5.4.1 insures that the central characters in the three categories $\mathbb{R}^f(G)$, $\mathcal{H}(G, \rho)$-mod and $\mathcal{H}'(Ψ)$-mod correspond. Using Theorem 5.3.3, one can relate the signature $K$-character for an admissible irreducible representation $π$ in $\mathbb{R}^f(G)$ with the signature $\mathcal{H}(K, \rho)$-character of $M_ρ(π)$ in $\mathcal{H}(G, \rho)$-mod, see section 5, particularly pages 32-33, in [BM1]. Via the isomorphism between $\mathcal{H}(K, \rho)$ and $\mathcal{H}_W$, in Definition 5.4.1 (ii), and the correspondence of tempered modules from Remark 5.4.2, the signature $\mathcal{H}(K, \rho)$-character of $M_ρ(π)$ is in turn identified with the $\mathcal{H}_W$-signature character of $\xi(M_ρ(π))$ in $\mathcal{H}'(Ψ)$-mod, cf. [BM1, Theorem 5.7]. Since the category $\mathcal{H}'(Ψ)$-mod (with a fixed elliptic central character) has Vogan’s property by Theorem 4.4.1, the claim follows. The argument is identical with the one in the proof of Theorem 1.1 of [BM1, page 33]. □

6. Unitary correspondences

In this section, we give two examples of unitary correspondences as in Theorem 5.5.2, and present certain applications to unitary equivalences with endoscopic groups.

6.1. Unramified principal series. Assume that $G$ is the $F$-points of a linear reductive algebraic group over $F$. Denote by $v_F$ the valuation function on $\mathbb{F}$. Standard references for the discussion about unramified principal series are [Car], [Bo1].

Fix $A$ a maximally split torus in $G$ and set

$$M = C_G(A), \quad W(G, A) = N_G(A)/M. \tag{6.1.1}$$

Let $X^*(M)$ and $X_*(M)$ denote the lattices of algebraic characters and cocharacters of $M$, respectively, and $(\lambda, \lambda)$ their natural pairing. Define the valuation map

$$v_M : M \rightarrow X_*(M), \quad (\lambda, v_M(m)) = v_F(\lambda(m)), \quad \text{for all} \ m \in M, \ \lambda \in X^*(M). \tag{6.1.2}$$

Set $0^M = \ker v_M$ and $Λ(M) = \text{Im} v_M$. Similarly, define $v_A$, $0^A$, and $Λ(A)$. Since $A$ is a torus, we have $Λ(A) = X_*(A)$. Moreover, we have $X_*(A) \subset Λ(M) \subset X_*(M)$. (Notice that $Λ(M) = X_*(M)$ precisely when $M = A$, i.e., $G$ is $F$-split.)

The group of unramified characters of $M$ (i.e., characters trivial on $0^M$) will be denoted by $\hat{M}^u$. For every character $\chi \in \hat{M}^u$, let $X(\chi)$ denote the corresponding unramified principal series. It is clear that

$$\hat{M}^u \cong \text{Hom}(Λ(M), \mathbb{C}^\times), \tag{6.1.3}$$

so if we define $T' = \text{Spec}\mathbb{C}[Λ(M)]$, a complex algebraic torus, we have a natural identification

$$\hat{M}^u = T'. \tag{6.1.4}$$

Let $K$ and $I$ be a special maximal compact open subgroup of $G$ and an Iwahori subgroup, respectively, attached, using the Bruhat-Tits building, to the torus $A$ and a special vertex $x_0$ (see [Ti], or [Car, section 3.5]). Then $0^A = A \cap K$ and $0^M = M \cap K$. The Weyl group $W(G, A)$ acts on $X_*(M)$ preserving $X_*(A)$ and $Λ(M)$. If we let $\hat{W}(G, A) = W(G, A) \ltimes Λ(M)$ denote the extended Weyl group, the Bruhat-Tits decomposition is

$$G = I \hat{W}(G, A) I \quad \text{and} \quad K = I W(G, A) I. \tag{6.1.5}$$

The subquotients of the minimal (unramified) principal series $X(\chi)$ have inertial support $1 = [A, 1_A]$, where $1_A$ denotes the trivial character on $A$. In other words, the irreducible subquotients of the minimal principal series form the irreducible objects of the category $\mathbb{R}^f(G)$. 
**Theorem 6.1.1** ([Bo1, Cas]). The pair $(\mathcal{I}, 1_\mathcal{I})$ is a 1-type (in the sense of Definition 5.3.1) for $G$, i.e., $\mathcal{R}^1_G = \mathcal{R}^1_{(\mathcal{I}, 1_\mathcal{I})}(G)$. Moreover, in the equivalence of categories with $\mathcal{H}(G, \mathcal{I})$, the tempered modules correspond.

In this case, the structure of $\mathcal{H}(G, 1_\mathcal{I})$ is well-known by [IM]. Its description with generators and relations and the explicit parameters are in the tables of [Ti]. In the terminology of section 2, it is an affine Hecke algebra $\mathcal{H}(\Psi)$ with certain unequal parameters of geometric type for a root datum $\Psi$. More precisely, with the notation from section 2, particularly Definition 2.1.1, we have $\Psi = (X, X', R, R')$, where

1. $X = X^*(T') (= \Lambda(M))$, $X' = X_s(T')$;
2. the Weyl group of $\Psi$ is $W(G, A)$;
3. $R'$ is the set of “restricted roots” of $A$ in $G$ (see [Ti, section 1.9] or [Car, page 141]).

This implies that $(\mathcal{I}, 1_\mathcal{I})$ is an affine 1-type in the sense of Definition 5.4.1. Thus we have:

**Theorem 6.1.2.** The categories $\mathcal{R}^1_G$ and $\mathcal{H}(G, 1_\mathcal{I})$ are unitarily equivalent.

**Remark 6.1.3.** Following Theorem 2.2.1, we see that the central characters of $\mathcal{H}(G, 1_\mathcal{I})$ are in one-to-one correspondence with $W(G, A)$-conjugacy classes in $T'$. We will use this fact in section 6.2.

### 6.2. Quasisplit groups

We retain the notation from 6.1. In this subsection we explain a correspondence between unitarizable principal series of a quasisplit, nonsplit, quasisimple $p$-adic group $G$ and certain endoscopic split groups. We assume in addition that $G$ splits over an unramified extension of $\mathbb{F}$. The key observation is that while $\mathcal{H}(G, 1_\mathcal{I})$ may have unequal parameters, all of the graded Hecke algebras attached can be identified naturally with graded Hecke algebras with equal parameters. For this we need to examine the Iwahori-Hecke algebra and its graded versions more closely. Assume that the root datum $\Psi$ for $\mathcal{H}(G, 1_\mathcal{I})$ is non-simply laced root datum, and with parameters $\lambda$ and $\lambda^*$, the latter occurring when $\Psi$ is of type $B$.

Let $\alpha_s, \alpha_\ell$ denote a short root and a long root respectively, and let $|\alpha|$ denote the squared length of a root $\alpha$. The following lemma can be verified by inspecting Tits' tables for quasisplit groups ([Ti]).

**Lemma 6.2.1.** The parameters of $\mathcal{H}(G, 1_\mathcal{I})$ satisfy the conditions:

1. $\frac{\lambda(\alpha_s)}{\lambda(\alpha_\ell)} \in \left\{ 1, \frac{|\alpha_s|}{|\alpha_\ell|} \right\}$;
2. $\frac{\lambda(\alpha_s) \pm \lambda^*(\alpha_s)}{2\lambda(\alpha_\ell)} \in \left\{ 0, \frac{|\alpha_s|}{|\alpha_\ell|}, 1 \right\}$.

These conditions guarantee that for every graded Hecke algebra $\mathbb{H}_{\mu_\sigma}$ that appears in Remark 2.3.3 (via Theorem 2.3.2), the parameters $\mu_\sigma(\alpha)$ satisfy one of the following properties:

(a) $\mu_\sigma(\alpha_s) = \mu_\sigma(\alpha_\ell)$,
(b) $\mu_\sigma(\alpha_s) = \frac{|\alpha_s|}{|\alpha_\ell|} \mu_\sigma(\alpha_\ell)$,
(c) $\mu_\sigma(\alpha_s) = 0$.

In case (a), $\mathbb{H}_{\mu_\sigma}$ is a graded Hecke algebra with equal parameters. In case (b), we have a natural isomorphism

$$\mathbb{H}_{\mu_\sigma} \cong \mathbb{H}_{\mu_\sigma}^{\mathcal{I}}$$

where $\mathbb{H}_{\mu_\sigma}^{\mathcal{I}}$ is the graded Hecke algebra attached to the dual root system to that of $\mathbb{H}_{\mu_\sigma}$ and with parameters $\mu_\sigma(\alpha_\ell) = \mu_\sigma(\alpha_s)$, $\mu_\sigma(\alpha_s) = \mu_\sigma(\alpha_s)$. Notice that $\mathbb{H}_{\mu_\sigma}^{\mathcal{I}}$ is a graded
Hecke algebra with equal parameters. In case (c), we have a natural isomorphism (see [BC, Proposition 4.6])
\[ H_{\mu_\sigma} \cong \mathbb{C}[W_\sigma] \times H^0_{\mu_\sigma}, \]  
(6.2.2)
where \( W_\sigma \) is the reflection subgroup of \( W \) generated by the simple short roots, and \( H^0_{\mu_\sigma} \) is the graded Hecke algebra (with equal parameter \( \mu_\sigma(\alpha_i) \)) corresponding to the root system of long roots.

Every unramified principal series \( X(\chi) \) contains a unique irreducible \((\mathcal{K},-)\)-spherical subquotient \( \overline{X}(\chi) \). It is well-known that two unramified principal series \( X(\chi) \) and \( X(\chi') \) have the same composition factors, and in particular \( \overline{X}(\chi) \cong \overline{X}(\chi') \) if and only if
\[ \chi = w\chi', \]
for some \( w \in W(G,A) \).

Let \( \widehat{G}_{\text{sph}} \) denote the set of isomorphism classes of irreducible spherical representations of \( G \). Therefore, we have a one-to-one correspondence
\[ \widehat{G}_{\text{sph}} \leftrightarrow W(G,A)\text{-orbits in } \hat{M}^u = T'. \]  
(6.2.3)
(A better way to express this correspondence is via the Satake isomorphism \( \mathcal{H}(\hat{G},\mathcal{K}) \cong \mathbb{C}[\hat{M}^0\hat{M}]^W(G,A) \), see for example [Car, Theorem 4.1].)

We need to recast this bijection in terms of the dual L-group. Let \( G \) denote the connected group dual (in the sense of Langlands) to \( G \), and let \( T \) be a maximal torus in \( G \). Let \( \Psi(G) = (X^*(T), X_*(T), R(G,T), R^\tau(G,T)) \) be the corresponding root datum. The inner class of \( G \) defines a homomorphism \( \tau : \Gamma \to \text{Aut}(\Psi(G)) \), where \( \Gamma = \text{Gal}(\overline{F}/F) \). Since we assumed that \( G \) is quasisimple unramified, we know that the image \( \tau(\Gamma) \subset \text{Aut}(\Psi(G)) \) is a cyclic group generated by an automorphism of order \( d \) (\( d \in \{2,3\} \)) which, by abuse of notation, we also call \( \tau \). We fix a choice of root vectors \( X_\alpha \), for \( \alpha \in R \). The automorphism \( \tau \) maps the root space of \( \alpha \) to the root space of \( \tau(\alpha) \). We normalize \( \tau \) such that
\[ \tau(X_\alpha) = X_{\tau(\alpha)}, \]
for all \( \alpha \). \((6.2.4)\)

This allows one to extend \( \tau \) to an automorphism of \( G \) in a canonical way.

**Definition 6.2.2.** Two elements \( x_1, x_2 \in G \) are called \( \tau \)-conjugate if there exists \( g \in G \) such that \( x_2 = g \cdot x_1 \cdot \tau(g^{-1}) \). For a subset \( S \subset G \), denote:
\[ N_G(S\tau) = \{ g \in G : g \cdot S \cdot \tau(g^{-1}) \subset S \}. \]

The construction of the L-group is such that we have (see [Bo2, section 6]):
\[ X^*(T') = X^*(T)^\tau, \quad W(G,A) = W(G,T)^\tau. \]  
(6.2.5)

In particular, we have an inclusion \( X^*(T') \hookrightarrow X^*(T) \), which gives a surjection \( \nu : T \to T' \). By [Bo2, Lemma 6.4], the map
\[ \nu' : T \times \langle \tau \rangle \to T', \quad \nu'((t, \tau)) = \nu(t), \]  
(6.2.6)
induces a bijection of \((N(T\tau), \tau)\)-conjugacy classes of elements in \( T \) and \( W(G,T)^\tau\)-conjugacy classes of elements in \( T' \).

**Theorem 6.2.3** (Langlands, cf. [Car, Theorem 3.1], [Bo2, Proposition 6.7]). There are bijective correspondences:
\[ \text{semisimple } \tau\text{-conjugacy classes in } G \quad \leftrightarrow \]
\[ (N_G(T\tau), \tau)\text{-conjugacy classes of elements in } T \quad \leftrightarrow \]
\[ W(G,T)^\tau\text{-orbits in } T' \longleftrightarrow \hat{G}_{\text{sph}}. \]  
(6.2.7)
Table 2. Nonsplit quasisplit unramified $\mathcal{G}$ and corresponding split $\mathcal{G}(\tau)$.

<table>
<thead>
<tr>
<th>Type of $\mathcal{G}$</th>
<th>Label in $[\text{Ti}]$</th>
<th>Order of $\tau$</th>
<th>Type of $\mathcal{G}(\tau)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{2n-1}$</td>
<td>$2A_{2n-1}$</td>
<td>2</td>
<td>$B_n$</td>
</tr>
<tr>
<td>$A_{2n}$</td>
<td>$2A_{2n}$</td>
<td>2</td>
<td>$C_n$</td>
</tr>
<tr>
<td>$D_{n}$</td>
<td>$2D_{n}$</td>
<td>2</td>
<td>$C_{n-1}$</td>
</tr>
<tr>
<td>$D_{4}$</td>
<td>$2D_{4}$</td>
<td>3</td>
<td>$G_2$</td>
</tr>
<tr>
<td>$E_6$</td>
<td>$2E_6$</td>
<td>2</td>
<td>$F_4$</td>
</tr>
</tbody>
</table>

**Definition 6.2.4.** Let $\pi$ be an irreducible representation in $\mathcal{N}^1(\mathcal{G})$. It occurs as a subquotient in an unramified principal series $X(\chi)$ with $\mathcal{K}$-spherical subquotient $\overline{X}(\chi)$. Via Theorem 6.2.3, to $\overline{X}(\chi)$ there corresponds a semisimple $\tau$-conjugacy class in $G$. We will refer to this class (or any representative of it) as the infinitesimal character of $\pi$.

Notice that Definition 6.2.4 is compatible, via Remark 6.1.3, to our conventions from section 2.

**Definition 6.2.5.** We fix now an elliptic element $s_e \in T^\tau$, and consider the subcategory $\mathcal{N}^1_{s_e}(\mathcal{G}) \subset \mathcal{N}^1(\mathcal{G})$ of representations of $\mathcal{G}$ with infinitesimal character having elliptic part $\tau$-conjugate to $s_e$.

Let

$$G(s_e\tau) = \{ g \in G : gs_e\tau(g^{-1}) = s_e \}$$

(6.2.8)

denote the twisted centralizer of $s_e$ in $G$. This is a potentially disconnected reductive group. Denote the identity component by $G(s_e\tau)_0$. Let $\mathcal{G}(s_e\tau)$ denote a split $p$-adic group whose Langlands dual is $G(s_e\tau)$. In particular, this means that the group is the split $\mathbb{F}$-points of a disconnected group whose identity component has root datum dual (in the sense of Langlands) to that of $G(s_e\tau)_0$, and whose group of components is isomorphic to $G(s_e\tau)/G(s_e\tau)_0$.

Using the analysis after Lemma 6.2.1, based on a case-by-case inspection of the tables in [Ti], we find that the graded algebra at $s_e$ of the affine Iwahori-Hecke algebras for $\mathcal{G}$ is naturally $\ast$-preserving isomorphic with the graded algebra at 1 of $\mathcal{G}(s_e\tau)$. When $s_e = 1$, we denote this group by $\mathcal{G}(\tau)$. We obtain the following consequence.

**Corollary 6.2.6.** The categories $\mathcal{N}^1_{s_e}(\mathcal{G})$ and $\mathcal{N}^1(\mathcal{G}(s_e\tau))$ are unitarily equivalent. In particular, there is a unitary equivalence for representations with Iwahori fixed vectors and real infinitesimal character between $\mathcal{G}$ and $\mathcal{G}(\tau)$.

**Example 6.2.7.** The explicit cases for quasisplit groups and real infinitesimal characters are in Table 2. For example, when $\mathcal{G}$ is the quasisplit adjoint unitary group in four variables ($\mathcal{G} = \text{PSU}(2, 2)$), we have $G = \text{SL}(4, \mathbb{C})$. The corresponding automorphism $\tau$ is

$$\tau(g) = J \cdot g \cdot J^t,$$

where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$.  

(6.2.9)

The torus $T$ is the diagonal one. If we choose $s_e = 1$, then $G(\tau) = \text{Sp}(4, \mathbb{C})$, so the correspondence of unitarity is with $\mathcal{G}(\tau) = \text{SO}(5, \mathbb{F})$, but if we choose $s_e$ diagonal $(1, 1, -1, -1)$, then $G(s_e\tau) = \text{SO}(4, \mathbb{C})$, and the correspondence of unitarity is with $\mathcal{G}(s_e\tau) = \text{SO}(4, \mathbb{F})$. 

6.3. Ramified principal series for split groups. A similar type of correspondence can be achieved for ramified principal series using the results of [Ro]. We assume that \( \mathcal{G} \) is split over \( \mathbb{F} \) and the same restrictions of the characteristic \( p \) as in [Ro, section 3]. Let \( \chi : A \to \mathbb{C}^\times \) be a ramified character and set \( \hat{\chi}_0 : \hat{0}A \to \mathbb{C}^\times, \hat{0} \chi = \chi|_{\hat{0}A} \). One considers the inertial class \( r = [A, \chi] \). This only depends on \( \hat{0} \chi \). The irreducible objects in \( \mathcal{H}^\dagger(\mathcal{G}) \) are the irreducible subquotients of minimal principal series \( X(\chi') \), where \( \hat{0} \chi' = \hat{0} \chi \).

In [Ro, section 3], a \( r \)-type \((J, \rho)\) is constructed, with \( \rho \) one dimensional, and the structure of the corresponding Hecke algebras is computed. Let \( G \) be the complex group dual to \( \mathcal{G} \). A semisimple element \( \hat{\chi}_0 \in G \) is attached to \( \hat{0} \chi \) [Ro, section 8]. We explain this construction next. Let \( \mathcal{W}_F \) be the Weil group and recall the short exact sequence

\[
1 \longrightarrow \mathcal{I}_F \longrightarrow \mathcal{W}_F \longrightarrow \mathbb{Z} \longrightarrow 1,
\]

where \( \mathcal{I}_F \) is the inertia group (open in \( \mathcal{W}_F \)), and the cyclic group \( \mathbb{Z} \) generated by a Frobenius element \( \text{Frob} : x \mapsto x^q \) in \( \text{Gal}(\mathbb{F}_q/\mathbb{F}_q) \). The Weil-Deligne group is \( \mathcal{W}_F' = \mathbb{C} \rtimes \mathcal{W}_F \), where \( w \in \mathcal{W}_F \) acts on \( \mathbb{C} \) by multiplication by \( ||w|| \), the norm of \( w \).

A Weil homomorphism \( \phi : \mathcal{W}_F' \to G \) is a continuous homomorphism satisfying certain properties (see for example [Bo2, §8.1]). In particular, \( \phi|_{\mathcal{W}_F} \) should consist of semisimple elements and \( \phi|_{\mathcal{C}} \) of unipotent elements. We say that \( \phi \) is unramified if \( \phi(\mathcal{I}_F) \) consists of central elements of \( G \).

Let \( \mathcal{W}_F^{ab} \) denote the abelian quotient of \( \mathcal{W}_F \). The Weil homomorphisms that parameterize minimal principal series descend to \( \mathbb{C} \rtimes \mathcal{W}_F^{ab} \). Let \( \mathcal{I}_F^{ab} \) denote the image of \( \mathcal{I}_F \) in \( \mathcal{W}_F^{ab} \). Recall that the reciprocity homomorphism \( \tau_F \) is an isomorphism of \( \mathcal{W}_F^{ab} \) onto \( \mathbb{F}^\times \), and induces an isomorphism of \( \mathcal{I}_F^{ab} \) onto \( \mathcal{O}^\times \).

To the character \( \hat{0} \chi : \hat{0}A \to \mathbb{C}^\times \), we attach a homomorphism

\[
\hat{\chi}_0 : \mathcal{I}_F^{ab} \to T \subset G.
\]

This is the unique homomorphism which makes the following diagram commutative for any \( \lambda \in X_*(A) = X^*(T) \):

\[
\begin{array}{ccc}
\mathcal{I}_F^{ab} & \xrightarrow{\hat{\chi}_0} & T \\
\downarrow{\tau_F} & & \downarrow{\lambda} \\
\mathbb{C}^\times & & \mathbb{C}^\times \\
\end{array}
\]

Now we consider

\[
\phi : \mathbb{C} \times \mathcal{W}_F^{ab} \to G \text{ such that } \phi|_{\mathcal{I}_F^{ab}} = \hat{\chi}_0.
\]

Such homomorphisms parameterize \( L \)-packets which have subquotients of minimal principal series \( X(\chi) \), with \( \chi|_{\hat{0}A} = \hat{0} \chi \). Define

\[
C_G(\hat{\chi}_0) = \text{centralizer in } G \text{ of the image of } \hat{\chi}_0.
\]

**Lemma 6.3.1** ([Ro, section 8]). \( C_G(\hat{\chi}_0) \) is the centralizer of a single semisimple element in \( G \).
By abuse of notation, we denote this semisimple element by \( \hat{\xi} \) too. Let \( \Psi_{\hat{\xi}} \) be the root datum for the identity component of the centralizer \( C_G(\hat{\xi}) \), and let \( \Gamma \) be the group of components of \( C_G(\hat{\xi}) \).

**Theorem 6.3.2** ([Ro, Theorem 6.3 and section 9]). The \( \tau \)-type \((J, \rho)\) is affine, and the Hecke algebra \( \mathcal{H}(G, \rho) \) is naturally isomorphic to the extended affine Hecke algebra \( \mathcal{H}(\Psi_{\hat{\xi}}) = \mathcal{H}(\Psi_{\hat{\xi}}) \times \Gamma \) with equal parameters.

**Corollary 6.3.3.** The categories \( \mathcal{R}^f(G) \) and \( \mathcal{H}(G, \rho)\)-mod (equivalently, \( \mathcal{H}(\Psi_{\hat{\xi}}) = \mathcal{H}(\Psi_{\hat{\xi}}) \times \Gamma\)-mod) are unitarily equivalent (in the sense of Definition 5.5.1).

By combining this with Theorem 6.1.2, we have an important consequence. Let \( G'(\hat{\xi}) \) be a split \( p \)-adic group dual to \( C_G(\hat{\xi}) \). The Iwahori Hecke algebra of this group is naturally identified with \( \mathcal{H}(\Psi_{\hat{\xi}}) \). By combining the previous corollary with Theorem 6.1.2, we have an important consequence.

**Corollary 6.3.4.** The categories \( \mathcal{R}^f(G) \) and \( \mathcal{R}^e(G'(0)) \) are unitarily equivalent via the equivalences of categories:

\[
\mathcal{R}^f(G) \cong \mathcal{H}(\Psi_{\hat{\xi}})\text{-mod} \cong \mathcal{R}^e(G'(0)).
\]

Assume now that \( G \) is adjoint. In this setting, [Re] gives the Deligne-Langlands-Lusztig classification for \( \mathcal{R}^f(G) \). Notice that any parameter \( \phi \) as in (6.3.4) has the image in fact in \( G(\hat{\xi}) \). Denote by \( \phi' \) the homomorphism obtained by restricting the range of \( \phi \) to \( G(\hat{\xi}) \). Then \( \phi' \) is, by definition, an unramified parameter for \( G(\hat{\xi}) \). Therefore, we have a one-to-one correspondence of L-packets

\[
\Psi : \phi \mapsto \phi', \quad \phi|_{I_{\rho}^{\text{ab}}} = \hat{\xi}, \quad \phi' \text{ unramified for } G(\hat{\xi}). \quad (6.3.6)
\]

The assumption that \( G \) is adjoint implies that \( G(\hat{\xi}) \) is connected. Following [Re], the correspondence (6.3.6) encodes the bijection between subquotients for the \( \hat{\xi} \)-ramified principal series of \( G(\hat{\xi}) \) and subquotients \( \phi' \) of the unramified principal series of \( G(\hat{\xi}) \). More precisely, let \( A_G(\phi) \) and \( A_{G(\hat{\xi})}(\phi') \) denote the component groups of the centralizers of the images of \( \phi \) and \( \phi' \) respectively. Let \( B^\phi_{G(\hat{\xi})} \) and \( B^{\phi'}_{G(\hat{\xi})} \) denote the varieties of Borel subgroups fixed by the images of \( \phi \) and \( \phi' \) respectively. We say that a representation of the component group is of Springer type if it appears in the action on the Borel-Moore homology of these varieties. Then there is a natural isomorphism

\[
A_G(\phi) \cong A_{G(\hat{\xi})}(\phi'), \quad (6.3.7)
\]

which induces a bijection \( \Psi \) of the component group representations of Springer type.

Then the reformulation of the corollaries in section 6.3, in the particular case when \( G \) is adjoint, is that the correspondence \( \Psi \) of (6.3.6) restricted to elements of Springer type, gives a one-to-one correspondence between hermitian and unitary representations, respectively.

**References**


(D. Barbasch) DEPT. OF MATHEMATICS, CORNELL UNIVERSITY, ITHACA, NY 14850
E-mail address: barbasch@math.cornell.edu

(D. Ciubotaru) DEPT. OF MATHEMATICS, UNIVERSITY OF UTAH, SALT LAKE CITY, UT 84112
E-mail address: ciubo@math.utah.edu