

Multiplicity matrices for the affine graded Hecke algebra

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Abstract

In this paper we are looking at the problem of determining the composition factors for the affine graded Hecke algebra via the computation of Kazhdan-Lusztig type polynomials. We review the algorithms of [9, 10], and use them in particular to compute, at every real central character which admits tempered modules, the geometric parameterization, the Kazhdan-Lusztig polynomials, the composition series, and the Iwahori-Matsumoto involution for the representations with Iwahori fixed vectors of the split p -adic groups of type G_2 and F_4 (and by the nature of the algorithms, for their Levi subgroups).

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1 Introduction

1.1

Let G be a complex, connected, simply-connected, semisimple Lie group, let H be a Cartan subgroup, and $B \supset H$ a Borel subgroup. Let \mathfrak{g} , \mathfrak{b} , \mathfrak{h} denote the respective Lie algebras, and let Δ , Δ^+ , Π be the corresponding roots, positive roots, and simple roots respectively, and let W be the Weyl group. If $\alpha \in \Delta$, the corresponding coroot is $\check{\alpha} \in \mathfrak{h}$, and the reflection in W is s_α . The pairing between \mathfrak{h} and \mathfrak{h}^* is denoted $\langle \cdot, \cdot \rangle$.

The *affine graded Hecke algebra* \mathbb{H} was introduced in [11]. We will only consider a special case of the definition, the “equal parameters” case. The generators of \mathbb{H} are the elements $\{t_{s_\alpha} : \alpha \in \Pi\}$ and $\{\omega : \omega \in \mathfrak{h}^*\}$. Here Π denotes the set of simple roots. As a \mathbb{C} -vector space,

$$\mathbb{H} = \mathbb{C}[W] \otimes \mathbb{A}, \quad (1.1.1)$$

where

$$\mathbb{A} = \text{Sym}(\mathfrak{h}^*). \quad (1.1.2)$$

The following commutation relations hold:

$$\omega t_{s_\alpha} = t_{s_\alpha} s_\alpha(\omega) + 2\langle \omega, \check{\alpha} \rangle, \quad \alpha \in \Pi, \omega \in \mathfrak{h}^*. \quad (1.1.3)$$

The center $Z(\mathbb{H})$ of \mathbb{H} consists of the W -invariants in \mathbb{A} ([11]):

$$Z(\mathbb{H}) = \mathbb{A}^W. \quad (1.1.4)$$

On any irreducible \mathbb{H} -module, which is necessarily finite dimensional, $Z(\mathbb{H})$ acts by a *central character*. Therefore, the central characters are parameterized by W -conjugacy classes in \mathfrak{h} . Let $\text{mod}_\chi(\mathbb{H})$ be the category of finite dimensional modules of \mathbb{H} with central character χ . This will be the main object of interest in this paper.

1.2

The connection with the representation theory of p -adic groups is well-known, and we briefly recall it next. Let \mathcal{G} be the split adjoint p -adic group whose dual group (in the sense of Langlands) is G . Let \mathcal{I} denote an Iwahori subgroup of \mathcal{G} . The *Iwahori-Hecke algebra*, denoted \mathcal{H} is the algebra of locally constant compactly supported \mathcal{I} -biinvariant functions under convolution. If a \mathcal{G} -representation (π, V) has \mathcal{I} -fixed vectors, then \mathcal{H} acts on $V^{\mathcal{I}}$. Let $C(\mathcal{I}, 1)$ be the category of admissible representations whose every subquotient is generated by its \mathcal{I} -fixed vectors.

Theorem 1 (Borel). *The association $V \mapsto V^{\mathcal{I}}$ is an equivalence of categories between $C(\mathcal{I}, 1)$ and the category $\text{mod}(\mathcal{H})$ of finite dimensional representations of \mathcal{H} .*

Casselman proved that every subquotient of a minimal principal series $I(\lambda)$ of \mathcal{G} , where λ is an unramified character of the maximal (split) torus of \mathcal{G} dual to H , is in $C(\mathcal{I}, 1)$, and in fact every irreducible object in $C(\mathcal{I}, 1)$ is of this form.

In [8], the Langlands (geometric) classification for the category $\text{mod}(\mathcal{H})$ is proved. By a result of Bernstein, the central characters of \mathcal{H} are parameterized by W -orbits of elements in H . If $\bar{\chi} \in H$, let $\text{mod}_{\bar{\chi}}(\mathcal{H})$ be the subcategory of modules with central character $\bar{\chi}$. The connection with the graded Hecke algebra is in [11]. The algebra \mathbb{H} is the associated graded object to a certain filtration in the Iwahori-Hecke algebra \mathcal{H} . Assume χ is a hyperbolic element of \mathfrak{h} , and then $\bar{\chi} = \exp(\chi)$ is a hyperbolic element of H . There is a natural correspondence between irreducible objects

$$\text{Irr mod}_{\bar{\chi}}(\mathcal{H}) \leftrightarrow \text{Irr mod}_{\chi}(\mathbb{H}). \quad (1.2.1)$$

We emphasize that, in (1.2.1), $\bar{\chi}$ is hyperbolic. If the category is $\text{mod}_s(\mathcal{H})$, with s arbitrary, one decomposes $s = s_e \cdot \bar{\chi}$ into the elliptic and hyperbolic parts. A similar correspondence holds, but the graded Hecke algebra in the right hand side is one for a root system defined by the centralizer of s_e in G (which is connected by Steinberg's theorem).

1.3

The geometric classification for $\text{mod}_{\chi}(\mathbb{H})$ exists as well ([12]; also [8, 11]), and we have standard modules X and irreducible quotients L . In the bijections of section 1.2, the standard, respectively irreducible, modules correspond.

The classification is expressed in terms of the geometry of the spaces $\mathfrak{g}_n(\chi)$:

$$G(\chi) = \{g \in G : \text{Ad}(g)\chi = \chi\}, \quad \mathfrak{g}_n(\chi) = \{y \in \mathfrak{g} : [\chi, y] = ny\}. \quad (1.3.1)$$

Let $\text{Orb}_n(\chi)$ denote the set of $G(\chi)$ orbits on $\mathfrak{g}_n(\chi)$. Assume that $n \in \mathbb{Z} \setminus \{0\}$. Then $\mathfrak{g}_n(\chi)$ is a prehomogeneous $G(\chi)$ -vector space ([7]), and in fact $\text{Orb}_n(\chi)$ is finite. For every $\mathcal{O} \in \text{Orb}_n(\chi)$, $\overline{\mathcal{O}} \setminus \mathcal{O}$ is the union of some orbits \mathcal{O}' with $\dim \mathcal{O}' < \dim \mathcal{O}$.

Theorem 2 ([12]). *The standard and irreducible objects in $\text{mod}_{\chi}(\mathbb{H})$ are in bijection with pairs $\xi = (\mathcal{O}, \mathcal{L})$, where*

1. \mathcal{O} is a $G(\chi)$ -orbit on $\mathfrak{g}_2(\chi)$;
2. \mathcal{L} is a $G(\chi)$ -equivariant local system on \mathcal{O} of Springer type.

More precisely, choose some $e \in \mathcal{O}$. Then \mathcal{L} corresponds to a representation ϕ of the component group $A(e, \chi) = G(e, \chi)/G(e, \chi)^0$. The representations ϕ which are allowed must be in the restriction from $A(e) = G(e)/G(e)^0$ to $A(e, \chi)$ of a representation which appears in the Springer correspondence.

1.4

In this setting, the Kazhdan-Lusztig conjectures take the following form.

Theorem 3 ([12]). *In (the Grothendieck group of) $\text{mod}_\chi(\mathbb{H})$:*

$$\begin{aligned} X_{\xi'} &= \sum_{\xi} P_{\xi, \xi'}(1) \cdot L_{\xi}, \quad \text{with} \\ P_{\xi, \xi'}(q) &= \sum_{i \geq 0} [\mathcal{L} : \mathcal{H}^{2i} IC(\overline{\mathcal{O}'}, \mathcal{L}') |_{\mathcal{O}}] \cdot q^i, \end{aligned} \quad (1.4.1)$$

where $\mathcal{H}^j IC()$ denotes the j -th cohomology sheaf of the intersection cohomology complex.

(In the setting of the affine Iwahori-Hecke algebra, the similar result was established in [3].)

Corollary 1. *In particular, $P_{\xi, \xi} = 1$, and if $\xi \neq \xi'$, then $P_{\xi, \xi'} = 0$, unless $\mathcal{O}' \subsetneq \overline{\mathcal{O}}$.*

1.5

An important feature of \mathbb{H} is the *Iwahori-Matsumoto involution* IM ,

$$IM(t_w) = (-1)^{\ell(w)} t_w, \quad w \in W, \quad IM(\omega) = -\omega, \quad \omega \in \mathfrak{h}^*. \quad (1.5.1)$$

which gives a bijection

$$IM : \text{Irr } \text{mod}_\chi \mathbb{H} \leftrightarrow \text{Irr } \text{mod}_{-\chi} \mathbb{H}. \quad (1.5.2)$$

One also has an obvious involution

$$\kappa : \mathbb{H} \rightarrow \mathbb{H}, \quad \kappa(w) = w, \quad \kappa(\omega) = -\omega. \quad (1.5.3)$$

Finally, there is the geometric Fourier-Deligne transform FD ([10], 2.1; [6]) which induces a bijection between irreducible $G(\chi)$ -equivariant local systems supported on orbits in $Orb_2(\chi)$ and irreducible $G(\chi)$ -equivariant local systems supported on orbits in $Orb_{-2}(\chi)$. The connection between these maps is given by [6].

Theorem 4. *In $\text{mod}_\chi(\mathbb{H})$, the Fourier-Deligne transform and the (modified) involution IM induce the same bijection on irreducible modules:*

$$\kappa \circ IM = FD. \quad (1.5.4)$$

1.6

The goal is to compute the matrix of multiplicities of theorem 3 for $\text{mod}_\chi(\mathbb{H})$. We will restrict ourselves (as we may by the theorems exposit in section 1.2) to the case when $\chi \in \mathfrak{h}$ is hyperbolic (“real”). We follow the algorithms presented in [9, 10]. In sections 2 and 3 we review in a combinatorial way these algorithms. We should mention that there is no resemblance between this “p-adic” algorithm, and the classical algorithms for computing Kazhdan-Lusztig polynomials for complex or real groups. One of the very particular features of this algorithm, at the same time its main difficulty, is the use of the Fourier-Deligne transform. As a byproduct of the calculations (and by theorem 4), one obtains a (difficult) procedure for computing the Iwahori-Matsumoto involution on the \mathcal{G} -modules in $\text{Irr } C(\mathcal{I}, 1)$. An explicit, “closed formula”, description of the action of IM on $\text{Irr } C(\mathcal{I}, 1)$ is only known for $GL(n)$, by [19, 13].

A second feature of the algorithm is that in order to carry out the calculation for G , one needs to have done this first for all Levi subgroups of G .

Let v denote an indeterminate, which in the end will be specialized to $v = 1$. We will only consider *real* central characters, i.e., W -conjugacy classes of hyperbolic semisimple elements in \mathfrak{h} . The output of the algorithm is a square matrix of size $\#\text{Irr } \text{mod}_\chi(\mathbb{H})$ with (polynomial) entries in $\mathbb{Z}[v]$. If $c_{\xi, \xi'}(v)$ is such an entry, then the relation with the (Kazhdan-Lusztig) polynomial $P_{\xi, \xi'}(q)$ from theorem 3 is

$$\varepsilon \cdot c_{\xi, \xi'}(v) = v^{\dim \mathcal{O}' - \dim \mathcal{O}} \cdot P_{\xi, \xi'}\left(\frac{1}{v^2}\right), \quad (1.6.1)$$

where $\varepsilon \in \{+1, -1\}$ is a sign depending only on ξ' .

In other words, the polynomials $c_{\xi, \xi'}$ computed by the algorithm are, up to sign, those which give conjecturally the degrees in the Jantzen filtration.

In section 4, we give some simple examples of how the algorithm is applied. The regular case, when $\chi = 2\tilde{\rho}$, so that the trivial module is in $\text{mod}_\chi(\mathbb{H})$, is geometrically trivial. We present it just as an illustration of the combinatorics of sections 2 and 3. For the same reason, we include some small examples which were known by different methods. The first such example is in $GL(4)$ ([17],[19]). In $GL(n)$, [18] relates the polynomials $P_{\xi, \xi'}(q)$ with Kazhdan-Lusztig polynomials in category \mathcal{O} , giving therefore a different, indirect, way to determine them. Secondly, we present the cases in $\mathfrak{g} = \mathfrak{sp}(6)$ and $\mathfrak{g} = G_2$ where there are cuspidal (in the sense of Lusztig) local systems. The multiplicity matrices and the action of the Iwahori-Matsumoto involution in these examples appear in [15]. The approach of [15] is different than that of [9, 10] and the calculations in this paper, and relies on an analysis of the endomorphism algebras of intertwining operators. In particular, this approach is very successful when χ has “rank one singularities”, i.e., the centralizer $G(\chi)$ is essentially $SL(2, \mathbb{C})$. This case is treated in detail in sections 9-13 of [15], and the $\mathfrak{sp}(6)$ and G_2 examples fall in this category. (See also [16], 13.2, for similar calculations when the Hecke algebra is of type F_4 with unequal parameters appearing in the classification of unipotent representations of split p -adic E_8 .)

In general, when the centralizer $G(\chi)$ is more complicated, the computation of intertwining operators is very difficult, and it would be interesting to have a general algorithm via the approach of [15].

In section 5, we calculate the polynomials when $\mathfrak{g} = F_4$, for all χ which are middle elements of nilpotent orbits. By the geometric classification, they are precisely the (hyperbolic) central characters which afford tempered modules. These are the difficult cases of the algorithm. The most interesting case is when χ is the middle element of the unique nilpotent orbit in F_4 which has a cuspidal local system. This is presented in more detail in section 5.1. Here $G(\chi)$, which is of type $A_1 \times A_2$, has 12 orbits on the 12-dimensional $\mathfrak{g}_2(\chi)$, and the centralizer in $G(\chi)$ of a point in the open orbit is the group S_4 . The split p -adic group of type F_4 has four Iwahori-spherical discrete series at this central character.

For parts of these calculations (most notably, Weyl group conjugations in F_4 , and checking if certain vectors are in the radical of the bilinear form defined in 2.5), I used a computer and a computer algebra system. In all the examples, the notation for nilpotent orbits is as in [2].

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2 Ingredients

We recall the algorithm in [9, 10] in a combinatorial language. The intention is to express all the elements of the algorithm purely in terms of the Weyl group and the roots in Δ . To this end, we will record two equivalent descriptions for the same object: the first labeled (*) as in [10], and the second, equivalent, but more combinatorial, labeled (**) (which often appears in [9] as well).

Throughout sections 2 and 3, we fix a semisimple element $\chi \in \mathfrak{h}$ such that χ is the middle element of a Lie triple in \mathfrak{g} . We assume that χ is dominant with respect to Δ^+ :

$$\langle \alpha, \chi \rangle \geq 0, \text{ for all } \alpha \in \Delta^+, \quad (2.0.2)$$

so that, in particular, $w_0\chi = -\chi$, where w_0 is the longest Weyl group element.

2.1

The assumption on χ is that there exists a Lie algebra homomorphism

$$\phi : sl(2, \mathbb{C}) \rightarrow \mathfrak{g}, \quad \phi\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = \chi. \quad (2.1.1)$$

By the representation theory of $sl(2, \mathbb{C})$, this implies that the element χ induces a grading on the Lie algebra \mathfrak{g} , by its ad -action:

$$\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n, \quad \mathfrak{g}_n = \{x \in \mathfrak{g} : [\chi, x] = nx\}. \quad (2.1.2)$$

For any subalgebra \mathfrak{p} of \mathfrak{g} , we will denote similarly $\mathfrak{p}_n = \mathfrak{g}_n \cap \mathfrak{p}$. We also define

$$\begin{aligned} r_n(\mathcal{R}) &= \{\alpha \in \mathcal{R} : \langle \alpha, \chi \rangle = n\}, \text{ for any subset of roots } \mathcal{R} \subset \Delta, \text{ and} \\ r_n(w) &= r_n(\Delta) \cap (w^{-1} \cdot \Delta^+), \text{ for any element } w \in W. \end{aligned} \quad (2.1.3)$$

Some immediate properties that we will need later are:

Lemma 1. *For all $w \in W$, $n \in \mathbb{Z}$:*

- (a) $r_n(w) = r_{-n}(w_0 w)$;
- (b) $r_n(w) \cap r_n(w_0 w) = \emptyset$;
- (c) $r_n(w) \cup r_n(w_0 w) = r_n(\Delta)$.

Proof. Straightforward. □

2.2

Define

$$\begin{aligned} G(\chi) &= \{g \in G : Ad(g)\chi = \chi\}, \\ W(\chi) &= \{w \in W : w\chi = \chi\}, \\ \mathcal{B}(\chi) &= G(\chi)\text{-orbits in } \{\mathfrak{b}' : \mathfrak{b}' \text{ Borel subalgebra with } \chi \in \mathfrak{b}'\}. \end{aligned} \quad (2.2.1)$$

The space in which the constructions will take place is $\mathcal{K}(\chi)$, defined as

$$\begin{aligned} (*) : \mathcal{K}(\chi) &= \mathbb{Q}(v)\text{-vector space with basis } \mathcal{B}(\chi), \\ (**) : \mathcal{K}(\chi) &= \mathbb{Q}(v)\text{-vector space with basis } W/W(\chi). \end{aligned} \quad (2.2.2)$$

The space $\mathcal{K}(\chi)$ has two involutions that we consider. The first is

$$\beta : \mathcal{K}(\chi) \rightarrow \mathcal{K}(\chi), \text{ determined by } \beta(v) = v^{-1}, \text{ and the identity on the } \mathbb{Q}(v)\text{-basis.} \quad (2.2.3)$$

The second involution σ associates to each Borel subalgebra, the opposite Borel subalgebra, and it is the identity on $\mathbb{Q}(v)$. On $W(\chi)$ -cosets, this is determined by

$$(**) : \sigma(w) = w_0 \cdot w \text{ and } \sigma(v) = v. \quad (2.2.4)$$

2.3

Consider the variety

$$\mathcal{S}_\chi = \{(\mathfrak{b}', \mathfrak{b}'') : \chi \in \mathfrak{b}' \cap \mathfrak{b}''\}, \quad (2.3.1)$$

with the natural diagonal action of $G(\chi)$. One defines the space $(\mathcal{B} \times \mathcal{B})(\chi)$:

$$(*) : (\mathcal{B} \times \mathcal{B})(\chi) = G(\chi)\text{-orbits on } \mathcal{S}_\chi, \text{ which can naturally be identified with} \quad (2.3.2)$$

$$(**) : (\mathcal{B} \times \mathcal{B})(\chi) = (W \times W)/W(\chi) \text{ (where } W(\chi) \text{ is regarded as the diagonal subgroup).}$$

This space is equipped with a function

$$\tau : (\mathcal{B} \times \mathcal{B})(\chi) \rightarrow \mathbb{Z} \quad (2.3.3)$$

as follows. For a Borel subalgebra \mathfrak{b}' , let \mathfrak{u}' denote the unipotent radical. Then

$$\begin{aligned} (*) : \tau((\mathfrak{b}', \mathfrak{b}'')) &= -\dim \frac{\mathfrak{u}'_0 + \mathfrak{u}''_0}{\mathfrak{u}'_0 \cap \mathfrak{u}''_0} + \dim \frac{\mathfrak{u}'_2 + \mathfrak{u}''_2}{\mathfrak{u}'_2 \cap \mathfrak{u}''_2}, \text{ or, equivalently,} \\ (**) : \tau((w_1, w_2)) &= \#(r_2(w_1) \vee r_2(w_2)) - \#(r_0(w_1) \vee r_0(w_2)), \end{aligned} \quad (2.3.4)$$

where \vee denotes the symmetric difference set operator. It is proved in [10] that τ is the same if one replaces 2 in the formulas above by -2 .

2.4

Set

$$c = \#r_2(\Delta) - \#r_0(\Delta), \quad (2.4.1)$$

where r_n is defined in (2.1.3).

Lemma 2. (a) *The map τ in definition (**) of (2.3.4) is well-defined, i.e.,*

$$\tau((w_1 w, w_2 w)) = \tau((w_1, w_2)), \text{ for any } w \in W(\chi).$$

(b) *Since $w_0 \chi = -\chi$, $\tau((w_1, w_2)) = \tau((w_1 w_0, w_2 w_0))$, for every $(w_1, w_2) \in W \times W$.*

(c) *For any $w_1, w_2 \in W$,*

$$\tau((w_1, w_2)) + \tau((\sigma(w_1), w_2)) = c.$$

(So this sum is independent of w_1, w_2).

Proof. Part (a) is immediate. For part (b), one uses lemma 1.(a) and the observation after the equation (2.3.4). Part (c) follows from lemma 1, parts (b) and (c). \square

2.5

Let

$$pr_j : (\mathcal{B} \times \mathcal{B})(\chi) \rightarrow \mathcal{B}(\chi), \quad j = 1, 2 \quad (2.5.1)$$

denote the projection onto the j -th coordinate. One defines a symmetric bilinear form on $\mathcal{K}(\chi)$

$$(\cdot) : \mathcal{K}(\chi) \times \mathcal{K}(\chi) \rightarrow \mathbb{Q}(v)$$

by

$$(*) : e_\chi^{-1} \cdot ([\mathbf{b}'] : [\mathbf{b}'']) = \sum_{\substack{\Omega \in (\mathcal{B} \times \mathcal{B})(\chi) \\ pr_1 \Omega = [\mathbf{b}'], pr_2 \Omega = [\mathbf{b}'']}} (-v)^{\tau(\Omega)}, \quad (2.5.2)$$

$$(**) : e_\chi^{-1} \cdot ([w_1] : [w_2]) = \sum_{[(w', w'')] \in pr_1^{-1}([w_1]) \cap pr_2^{-1}([w_2])} (-v)^{\tau((w', w''))} \quad (2.5.3)$$

$$= \sum_{w \in [w_1]} (-v)^{\tau((w, w_2))} = \sum_{w \in [w_2]} (-v)^{\tau((w_1, w))}.$$

In these formulas, $[\bullet]$ denotes the class of an element \bullet in $\mathcal{B}(\chi)$, and similarly in $(\mathcal{B} \times \mathcal{B})(\chi)$.

The factor $e_\chi \in \mathbb{Q}(v)$ is a normalization factor, and it depends only on χ . The choice that we will use is $e_\chi = (1 - v^2)^{-\text{rk } \mathfrak{g}}$ ([9]), so that we have the identity

$$\beta((\beta(\xi), \beta(\xi'))) = (-1)^{\text{rk } \mathfrak{g}} (-v)^{2 \text{rk } \mathfrak{g} - c} (\sigma(\xi) : \xi'), \text{ for all } \xi, \xi' \in \mathcal{K}(\chi). \quad (2.5.4)$$

In general, the bilinear form $(:)$ is degenerate. Let Rad denote its radical. From equation (2.5.4), we see that

$$\beta(Rad) = Rad. \quad (2.5.5)$$

Proposition 1 ([10]). *The dimension of $\mathcal{K}(\chi)/Rad$ equals $\#\text{Irr mod}_\chi(\mathbb{H})$, the number of inequivalent, irreducible representations with central character χ .*

2.6 Examples

To illustrate the definitions so far, we give some examples. In the following tables, for simplicity, we will take the normalization factor $e_\chi = 1$.

2.6.1 A_2

The simple roots are $\{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3\}$, and let the simple reflections be denoted by s_1 and s_2 . Then $W = \{1, s_1, s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1\}$. We look at $\chi = 2\check{\rho} = (2, 0, -2)$. The bilinear form is

	1	$s_1 s_2 s_1$	$s_1 s_2$	$s_2 s_1$	s_1	s_2
1	1	v^2	$-v$	$-v$	$-v$	$-v$
$s_1 s_2 s_1$	v^2	1	$-v$	$-v$	$-v$	$-v$
$s_1 s_2$	$-v$	$-v$	1	v^2	v^2	1
$s_2 s_1$	$-v$	$-v$	v^2	1	1	v^2
s_1	$-v$	$-v$	v^2	1	1	v^2
s_2	$-v$	$-v$	1	v^2	v^2	1

In this case, $\dim(Rad) = 2$, and a basis is given by $\{s_1 - s_2 s_1, s_2 - s_1 s_2\}$. This is a particular case of section 4.1.

2.6.2 C_2 , $\chi = (1, 1)$.

The simple roots are $\{\epsilon_1 - \epsilon_2, 2\epsilon_2\}$. We look at $\chi = (1, 1)$, the middle nilpotent element of the nilpotent orbit (22) in $\mathfrak{sp}(4, \mathbb{C})$. Then $W(\chi) = \{1, s_1\}$, and $W/W(\chi) = \{[1], [s_2], [s_1 s_2], [s_2 s_1 s_2]\}$. The bilinear form is

	$[1]$	$[s_2]$	$[s_1 s_2]$	$[s_2 s_1 s_2]$
$[1]$	$1 + v^{-2}$	$-v^{-1} - v$	$1 + v^2$	$-v - v^3$
$[s_2]$	$-v^{-1} - v$	2	$-2v$	$1 + v^2$
$[s_1 s_2]$	$1 + v^2$	$-2v$	2	$-v^{-1} - v$
$[s_2 s_1 s_2]$	$-v - v^3$	$1 + v^2$	$-v^{-1} - v$	$1 + v^{-2}$

In this example, the form is nondegenerate.

2.7

We have defined the space $\mathcal{K}(\chi)$ equipped with a degenerate bilinear form. The final basic ingredient of the algorithm is an *induction* map. Let \mathfrak{p} be a parabolic subalgebra of \mathfrak{g} , such that $\chi \in \mathfrak{p}$. The parabolic \mathfrak{p} is not necessarily standard with respect to the fixed Borel \mathfrak{b} (i.e., the choice of positive roots Δ^+). Let \mathfrak{l} denote the Levi component of \mathfrak{p} , and let \mathfrak{u}_p be the nilradical.

One can define $\mathcal{K}_\mathfrak{l}(\chi)$, $W_\mathfrak{l}(\chi)$ etc. similarly to the definitions for \mathfrak{g} in the previous sections. Let $proj : \mathfrak{p} \rightarrow \mathfrak{l}$ denote the projection onto the Levi factor.

The induction map is defined as

$$(*) : \text{ind}_{\mathfrak{p}}^{\mathfrak{g}} : \mathcal{B}_\mathfrak{l}(\chi) \rightarrow \mathcal{B}(\chi), \text{ind}_{\mathfrak{p}}^{\mathfrak{g}}(\mathfrak{b}') = proj^{-1}(\mathfrak{b}'). \quad (2.7.1)$$

One can define the same map in terms of the Weyl group. The roots in \mathfrak{u}_p are a subset of Δ , but not necessarily of Δ^+ . Let w_p be a Weyl group of minimal length such that $w_p(\Delta(\mathfrak{u}_p)) \subset \Delta^+$. Then

$$(**) : \text{ind}_{\mathfrak{p}}^{\mathfrak{g}} : \mathcal{B}_\mathfrak{l}(\chi) \rightarrow \mathcal{B}(\chi), \text{ind}_{\mathfrak{p}}^{\mathfrak{g}}([w]) = [w \cdot w_p^{-1}]. \quad (2.7.2)$$

3 Bases

The goal is to construct two pairs of bases $(\mathcal{Z}_+, \mathcal{U}_+)$ and $(\mathcal{Z}_-, \mathcal{U}_-)$ for $\mathcal{K}(\chi)/Rad$. The definition is inductive. The construction of the bases $\mathcal{Z}_+, \mathcal{U}_+$, respectively $\mathcal{Z}_-, \mathcal{U}_-$ is done in parallel in the space $\mathcal{K}(\chi)$, so we will use the subscript \pm for simplicity when there is no risk of confusion.

In the end, the change of bases matrix for the pair $(\mathcal{Z}_+, \mathcal{U}_+)$ (equivalently for $(\mathcal{Z}_-, \mathcal{U}_-)$) is the desired *multiplicity matrix*.

3.1

The standard modules with central character χ (and so the sets \mathcal{Z}_\pm and \mathcal{U}_\pm) are parameterized, as in theorem 2, by $Orb_{\pm 2}(\chi)$, the $G(\chi)$ -orbits on $\mathfrak{g}_{\pm 2}$ and local systems. [9] gives a parameterization of the orbits in terms of certain parabolic

subalgebras of \mathfrak{g} . We recall next the parameterization of orbits in \mathfrak{g}_2 . (The case of \mathfrak{g}_{-2} is absolutely analogous.) Let e be a (nilpotent) representative of an orbit $\mathcal{O} = \mathcal{O}_e$ of $G(\chi)$ in \mathfrak{g}_2 .

By the graded version of the Jacobson-Morozov triple ([9]), $e \in \mathfrak{g}_2$ can be embedded into a Lie triple $\{e, h, f\}$, such that $h \in \mathfrak{h} \subset \mathfrak{g}_0$, and $f \in \mathfrak{g}_{-2}$. From the pair of semisimple elements χ and h , one can define two associated parabolic subalgebras \mathfrak{p}_\pm as in [9].

Define a gradation of \mathfrak{g} with respect to h as well,

$$\mathfrak{g}^r = \{y \in \mathfrak{g} : [h, y] = ry\}, \quad r \in \mathbb{Z}, \quad (3.1.1)$$

and set

$$\mathfrak{g}_t^r = \mathfrak{g}_t \cap \mathfrak{g}^r. \quad (3.1.2)$$

Then

$$\mathfrak{g} = \bigoplus_{t,r \in \mathbb{Z}} \mathfrak{g}_t^r. \quad (3.1.3)$$

Set

$$\begin{aligned} \mathfrak{l} &= \bigoplus_{t=r} \mathfrak{g}_t^r, & \mathfrak{u}_- &= \bigoplus_{t < r} \mathfrak{g}_t^r, & \mathfrak{u}_+ &= \bigoplus_{t > r} \mathfrak{g}_t^r, \\ \mathfrak{p}_+ &= \mathfrak{l} \oplus \mathfrak{u}_+, & \mathfrak{p}_- &= \mathfrak{l} \oplus \mathfrak{u}_-. \end{aligned} \quad (3.1.4)$$

Since we want to emphasize the nilpotent element e , we will write in this section $\mathfrak{p}^e = \mathfrak{p}_+$ and similarly $\mathfrak{l}^e, \mathfrak{u}^e$. Clearly, $\mathfrak{h} \subset \mathfrak{g}_0^0 \subset \mathfrak{l}^e$.

Definition 1. *One says that χ is rigid for a Levi subalgebra \mathfrak{l} , if χ is congruent modulo the center $\mathfrak{z}(\mathfrak{l})$ to a middle element of a nilpotent orbit in \mathfrak{l} .*

We record the important properties of \mathfrak{p}^e . The centralizer of an element t in a group Q will be denoted below by $Z_Q(t)$, and its group of components by $A_Q(t)$.

Proposition 2 ([9]). *Consider the subalgebra \mathfrak{p}^e defined by (3.1.4), and let P^e be the corresponding parabolic subgroup.*

1. \mathfrak{p}^e depends only on e and not on the entire Lie triple $\{e, h, f\}$.
2. χ is rigid for \mathfrak{l}^e .
3. e is an element of the open $L^e(\chi)$ -orbit in \mathfrak{l}_2^e .
4. The $P(\chi)^e$ -orbit of e in \mathfrak{p}_2^e is open, dense in \mathfrak{p}^e .
5. $Z_{G(\chi)}(e) \subset P^e$.
6. The inclusion $Z_{L^e(\chi)}(e) \subset Z_{G(\chi)}(e)$ induces an isomorphism of the component groups

$$A_{L^e(\chi)}(e) \cong A_{G(\chi)}(e). \quad (3.1.5)$$

Remark. Note that, since χ is rigid in \mathfrak{l}^e , the component group $A_{L^e(\chi)}(e)$ is the component group corresponding to a nilpotent orbit in \mathfrak{l}^e , and these are all well-known (see [2]). In conclusion, part (6) of proposition 2 gives an effective way to compute the component groups for the orbits in $Orb_2(\chi)$.

In addition, an immediate corollary of (4) and (5) in proposition 2 is a dimension formula for the orbits in $Orb_2(\chi)$.

Corollary 2 ([10]). *For an orbit $\mathcal{O}_e \in Orb_2(\chi)$,*

$$\dim \mathcal{O}_e = \dim \mathfrak{p}_2^e - \dim \mathfrak{p}_0^e + \dim \mathfrak{g}_0, \quad (3.1.6)$$

where $\mathfrak{p}_i^e = \mathfrak{p}^e \cap \mathfrak{g}_i$, $i = 0, 2$.

Definition 2. *A parabolic subgroup P with Lie algebra \mathfrak{p} is called good for χ if $\mathfrak{p} = \mathfrak{p}^e$ for some nilpotent $e \in \mathfrak{g}_2$ (notation as in (3.1.4)), and such that it satisfies (2) in proposition 2.*

Let $\mathcal{P}(\chi)$ denote the set of good parabolic subgroups for χ . The parameterization of $Orb_2(\chi)$ is as follows.

Theorem 5 ([9]). *The map $\mathcal{O}_e \mapsto P^e$ defined above induces a bijection between $Orb_2(\chi)$ and $G(\chi)$ -conjugacy classes in $\mathcal{P}(\chi)$.*

3.2

Perhaps, it is better to think that $Orb_{\pm 2}(\chi)$ are parameterized by a set of pairs (\mathfrak{p}, h) , where h is a middle element of a nilpotent orbit in \mathfrak{l} , and \mathfrak{p} is a good parabolic. Let us call $\mathcal{E}(\chi)$ this parameter set.

In general, for computations, we will apply the following equivalent (but inelegant) procedure to determine $\mathcal{E}(\chi)$. Let \mathfrak{l} be a standard Levi subalgebra (corresponding to a subset of the simple roots Π), and let h be a middle element of a Lie triple $\{e, h, f\}$ for \mathfrak{l} , assumed dominant for $\Delta^+(\mathfrak{l})$. For a given \mathfrak{g} , there are finitely many pairs (\mathfrak{l}, h) like this. Let $\mathfrak{z}(e, h, f)$ denote the centralizer of $\{e, h, f\}$ in \mathfrak{g} . If

$$(**) : \text{there exists } w \in W \text{ such that } w(h + \nu_0) = \chi \text{ for some } \nu_0 \in \mathfrak{z}(e, h, f) \cap \mathfrak{h}, \quad (3.2.1)$$

then we set

$$s = w \cdot h, \quad (3.2.2)$$

and we define the two parabolic subalgebras $\mathfrak{p}_{s,-}$ and $\mathfrak{p}_{s,+}$ associated to s, χ as in (3.1.4), with common Levi subalgebra $\mathfrak{l}_s = \mathfrak{p}_{s,+} \cap \mathfrak{p}_{s,-}$. The pair $(\mathfrak{p}_{s,+}, s)$ parameterizes an orbit in $Orb_2(\chi)$, and similarly $(\mathfrak{p}_{s,-}, s)$ parameterizes an orbit in $Orb_{-2}(\chi)$. This is how all the orbits are indexed, in other words, our sets $\mathcal{E}(\chi)$ are formed of such pairs $(s, \mathfrak{p}_{s,\pm})$. Note, that by corollary 3.1, one can compute the dimension of the associated orbit at once.

3.3

We retain the notation from the previous subsections.

Consider $\mathcal{O} \in Orb_{\pm 2}(\chi)$, and let $\mathfrak{l}_s, \mathfrak{p}_{s,+}, \mathfrak{p}_{s,-}$ be the corresponding subalgebras defined in section 3.2. The bases \mathcal{Z}_{\pm} and \mathcal{U}_{\pm} are partitioned as:

$$\mathcal{Z}_{\pm} = \sqcup_{(s, \mathfrak{p}_{s, \pm}) \in \mathcal{E}(\chi)} \mathcal{Z}_{\pm}(\mathcal{O}), \quad \mathcal{U}_{\pm} = \sqcup_{(s, \mathfrak{p}_{s, \pm}) \in \mathcal{E}(\chi)} \mathcal{U}_{\pm}(\mathcal{O}). \quad (3.3.1)$$

If $\mathfrak{l}_s = \mathfrak{g}$, then $\mathcal{O} = \mathcal{O}_m$ is necessarily the unique maximal nilpotent orbit in $Orb_{\pm 2}(\chi)$.

Assume that $\mathcal{O} \neq \mathcal{O}_m$. Then \mathfrak{l}_s is a proper Levi subalgebra of \mathfrak{g} . By construction, there exists a Lie triple (e', s, f') of \mathcal{O} , such that $(e', s, f') \subset \mathfrak{l}_s$. Let $\mathcal{O}_{\mathfrak{l}_s}$ denote the nilpotent orbit of e' in \mathfrak{l}_s . By induction, we can assume that the bases $\mathcal{Z}_{\pm}^{\mathfrak{l}_s}(\mathcal{O}_{\mathfrak{l}_s})$ corresponding to central character s are constructed for \mathfrak{l}_s .

Then

$$\mathcal{Z}_+(\mathcal{O}) = \text{ind}_{\mathfrak{p}_{s,+}}^{\mathfrak{g}}(\mathcal{Z}_+^{\mathfrak{l}_s}(\mathcal{O}_{\mathfrak{l}_s})), \quad \mathcal{Z}_-(\mathcal{O}) = \text{ind}_{\mathfrak{p}_{s,-}}^{\mathfrak{g}}(\mathcal{Z}_-^{\mathfrak{l}_s}(\mathcal{O}_{\mathfrak{l}_s})). \quad (3.3.2)$$

We recall that the elements in each set $\mathcal{Z}_{\pm}(\mathcal{O})$ are parameterized by certain local systems, or equivalently certain representations of the group of components $\widehat{A}_G(\chi, e)$.

Proposition 3 ([10], 2.17). *If $\mathcal{O} \neq \mathcal{O}'$, and if $(\xi, \xi') \in \mathcal{Z}_+(\mathcal{O}) \times \mathcal{Z}_+(\mathcal{O}')$ or $(\xi, \xi') \in \mathcal{Z}_-(\mathcal{O}) \times \mathcal{Z}_-(\mathcal{O}')$, then*

$$(\xi : \xi') = 0. \quad (3.3.3)$$

3.4

Let us denote

$$\mathcal{Z}'_{\pm} = \mathcal{Z}_{\pm} \setminus \mathcal{Z}_{\pm}(\mathcal{O}_m), \quad \mathcal{U}'_{\pm} = \mathcal{U}_{\pm} \setminus \mathcal{U}_{\pm}(\mathcal{O}_m). \quad (3.4.1)$$

The multiplicity matrix computed by the algorithm is a matrix with coefficients in $\mathbb{Z}[v]$,

$$\mathcal{N} = \left(\begin{array}{c|c} \mathcal{N}_{1,1} & \mathcal{N}_{1,2} \\ \hline \mathcal{N}_{2,1} & \mathcal{N}_{2,2} \end{array} \right), \quad (3.4.2)$$

where

1. $\mathcal{N}_{1,1}$ is an upper unitriangular matrix of size $\#\mathcal{Z}'_{\pm} \times \#\mathcal{Z}'_{\pm}$ which will be computed in equation (3.4.7),
2. $\mathcal{N}_{1,2}$ is a matrix of size $\#\mathcal{Z}'_{\pm} \times \#\mathcal{Z}_{\pm}(\mathcal{O}_m)$ computed in equation (3.5.5),
3. $\mathcal{N}_{2,1}$ is the zero matrix of size $\#\mathcal{Z}_{\pm}(\mathcal{O}_m) \times \#\mathcal{Z}'_{\pm}$,
4. $\mathcal{N}_{2,2}$ is the identity matrix of size $\#\mathcal{Z}_{\pm}(\mathcal{O}_m) \times \#\mathcal{Z}_{\pm}(\mathcal{O}_m)$.

The sets \mathcal{Z}'_{\pm} were constructed by induction in section 3.3. One sets a partial ordering \leq on \mathcal{Z}'_{\pm} given by the dimensions of the corresponding orbits. In this order, the unique element in $\mathcal{Z}_{\pm}(0)$ is the minimal element.

Now we explain the construction of \mathcal{U}'_{\pm} . Define the matrices

$$\mathcal{M}_{\pm} = ((\xi : \xi'))_{\xi, \xi' \in \mathcal{Z}'_{\pm}}. \quad (3.4.3)$$

By proposition 3, these matrices are block-diagonal, with blocks of sizes $\#\mathcal{Z}_{\pm}(\mathcal{O})$.

Lemma 3 ([10], 1.11, 3.7). *The matrices \mathcal{M}_{\pm} are invertible.*

For every $\xi \in \mathcal{Z}'_{\pm}$, we find the vector

$$V_{\xi} = (a'_{\xi, \xi'})_{\xi' \in \mathcal{Z}'_{\pm}} = \mathcal{M}_{\pm}^{-1} \cdot ((\beta(\xi), \xi'))_{\xi' \in \mathcal{Z}'_{\pm}}. \quad (3.4.4)$$

By lemma 1.13 in [10], $a'_{\xi, \xi} = 1$, and $a'_{\xi, \xi'} = 0$ unless $\xi' \leq \xi$. Moreover, from [10], 1.14,

$$\beta(V_{\xi}^T) \cdot V_{\xi'} = \begin{cases} 1, & \text{if } \xi = \xi' \\ 0, & \text{if } \xi \neq \xi', \end{cases} \quad (3.4.5)$$

where V^T denotes the transpose of V .

Proposition 4 ([10]). *There exists a unique family $\{c_{\xi, \xi'} : \xi, \xi' \in \mathcal{Z}'_{\pm}\}$ such that*

(i) $c_{\xi, \xi} = 1$, $c_{\xi, \xi'} = 0$ if $\xi' \not\leq \xi$, and $c_{\xi, \xi'} \in v\mathbb{Z}[v]$ if $\xi' < \xi$;

(ii) $c_{\xi, \xi'} = \sum_{\xi'' \in \mathcal{Z}'_{\pm}} \beta(c_{\xi, \xi''}) a'_{\xi'', \xi'}$.

Set

$$\mu_{\xi} = \sum_{\xi' \in \mathcal{Z}'_{\pm}} c_{\xi, \xi'} \xi'. \quad (3.4.6)$$

Then $\mathcal{U}'_{\pm} = \{\mu_{\xi} : \xi \in \mathcal{Z}'_{\pm}\}$.

In other words, in the multiplicity matrix,

$$\mathcal{N}_{1,1} = (c_{\xi, \xi'})_{\xi, \xi' \in \mathcal{Z}'_{\pm}}. \quad (3.4.7)$$

3.5

It remains to explain the computation of the sets $\mathcal{Z}_+(\mathcal{O}_m)$ and $\mathcal{U}_+(\mathcal{O}_m)$. (The other pair is computed in the obvious analogue way.)

Since $\mathcal{K}(\chi)$ has a symmetric bilinear form, for every subspace $\mathcal{W} \subset \mathcal{K}(\chi)$, we can define the orthogonal complement \mathcal{W}^{\perp} . Clearly, $\text{Rad} \subset \mathcal{W}^{\perp}$.

Let \mathcal{W}_+ be the subspace spanned by \mathcal{Z}'_+ . In fact, \mathcal{Z}'_+ is a basis of \mathcal{W}_+ . Define the projections Y_+ , respectively Y_+^{\perp} of $\mathcal{K}(\chi)$ onto \mathcal{W}_+ , respectively \mathcal{W}_+^{\perp} . Explicitly,

$$Y_+^{\perp}(x) = x - Y_+(x), \quad (3.5.1)$$

where

$$Y_+(x) = \sum_{\xi \in \mathcal{Z}'_+} a_{x,\xi} \xi, \quad \text{and } (a_{x,\xi})_{\xi \in \mathcal{Z}'_+} = \mathcal{M}_+^{-1} \cdot ((x : \xi'))_{\xi' \in \mathcal{Z}'_+}. \quad (3.5.2)$$

Proposition 5 ([10]). *Let J_- be defined by*

$$J_- = \{\xi_0 \in \mathcal{Z}'_- : Y_+^\perp(\mu_{\xi_0}) \notin \text{Rad}\}. \quad (3.5.3)$$

The sets $\mathcal{Z}_+(\mathcal{O}_m)$ and $\mathcal{U}_+(\mathcal{O}_m)$ are then obtained as follows:

$$\mathcal{Z}_+(\mathcal{O}_m) = \{\xi = Y_+^\perp(\mu_{\xi_0}) : \xi_0 \in J_-\}, \quad \mathcal{U}_+(\mathcal{O}_m) = \{\mu_\xi = \mu_{\xi_0} : \xi_0 \in J_-\}. \quad (3.5.4)$$

This concludes the construction of the bases. To complete the matrix of multiplicities, one finds

$$\mathcal{N}_{1,2} = (c_{\xi,\xi'})_{\xi \in \mathcal{Z}_+(\mathcal{O}_m), \xi' \in \mathcal{Z}'_+} = \mathcal{M}_+^{-1} \cdot ((\mu_\xi : \xi''))_{\xi \in \mathcal{Z}_+(\mathcal{O}_m), \xi'' \in \mathcal{Z}'_+}. \quad (3.5.5)$$

Remarks.

(1) The transformation Y_+^\perp encodes the Fourier-Deligne transform FD (see [10]), and the essential fact in the construction of the proposition is that the FD dual of a local system on the open orbit in \mathfrak{g}_2 is a local system which *does not* live on the open orbit in \mathfrak{g}_{-2} . By theorem 4, the equivalent representation theoretic statement is that the Iwahori-Matsumoto dual of a tempered module is not tempered.

(2) Always, the basis element corresponding to the zero orbit in \mathcal{Z}'_- , $\xi_{triv} \in \mathcal{Z}'_-$, is in J_- . In fact, $\xi = Y_+^\perp(\xi_{triv}) \in \mathcal{Z}_+(\mathcal{O}_m)$ corresponds to the trivial local system on \mathcal{O}_m . The equivalent, representation theoretic statement is a combination of two facts: firstly, that the Iwahori-Matsumoto involution of the generic module is the spherical module, and secondly, that the generic module is parameterized by the trivial local system on \mathcal{O}_m ([1, 14]).

(3) To compute FD in general (not just for the elements supported on the open orbit), one can use the following procedure. Assume $\xi \in \mathcal{Z}_+$ corresponds to $(\mathcal{O}, \mathcal{L})$ and $\xi' \in \mathcal{Z}_-$ corresponds to $(\mathcal{O}', \mathcal{L}')$. Then $FD(\mathcal{L}) = \mathcal{L}'$ if and only if $\mu_\xi \in \mathcal{U}_+$ is the (unique) element of \mathcal{U}_+ such that

$$\{Y^+(\mu_{\xi'})\} \cup (\mathcal{U}_+ \setminus \{\mu_\xi\}) \quad (3.5.6)$$

is a linear independent set (actually a basis) of $\mathcal{K}(\chi)/\text{Rad}$.

4 Examples: the regular case, $gl(4)$, $sp(4)$, $sp(6)$, and G_2

In the explicit examples in $gl(4)$, $sp(4)$, $sp(6)$, G_2 , the symbol used to denote the $G(\chi)$ -orbits on \mathfrak{g}_2 and the local systems encodes the dimension of the orbit. When there are more orbits with the same dimension, we add an subscript a, b, \dots . If the component group is not trivial, then in these examples it is always $\mathbb{Z}/2\mathbb{Z}$, and we add a subscript t or s corresponding to the trivial, respectively the sign representations.

4.1 Regular central character

Recall that $\Pi \subset \Delta^+$ denotes the set of simple roots, and fix root vectors X_α . When $\chi = 2\check{\rho}$, the orbits and Kazhdan-Lusztig polynomials have an especially simple form:

$$\begin{aligned} \mathfrak{g}_2(\chi) &= \bigoplus_{\alpha \in \Pi} \mathbb{C} \cdot X_\alpha, \\ G(\chi) &= \text{the Cartan subgroup } H. \end{aligned} \tag{4.1.1}$$

There is a one-to-one correspondence

$$\text{Orb}_2(\chi) \leftrightarrow 2^\Pi, \tag{4.1.2}$$

where to every $\Pi_M \subset \Pi$ we associate the orbit $\mathcal{O}_M = \sum_{\alpha \in \Pi_M} \mathbb{C}^* \cdot X_\alpha$. All the orbits have smooth closures, and only trivial local systems appear, and therefore, all Kazhdan-Lusztig polynomials are either 0 or 1, depending on the closure ordering. The closure ordering is given by the inclusion of subsets of Π .

We include however the combinatorial calculation using the algorithm explained in section 2 and 3, just for the purpose to illustrate the elements of this algorithm.

If $\chi = 2\check{\rho}$, then $r_2(\chi, \Delta) = \Pi$ and $r_0(\chi, \Delta) = \emptyset$.

Let Π_M be a subset of Π , and $w_0(M)$ be the longest Weyl element in $W(M)$.

Lemma 4.

- (1) If $\Pi_M \subset \Pi$, then $r_2(\chi, w_0(M)) = \Pi \setminus \Pi_M$.
- (2) For any $\Pi_{M_1}, \Pi_{M_2} \subset \Pi$,

$$\tau(w_0(M_1) : w_0(M_2)) = |\Pi_{M_1} \vee \Pi_{M_2}|.$$

In particular, $\tau(w_0(M), 1) = |\Pi_M|$ and $\tau(w_0(M), w_0) = |\Pi| - |\Pi_M|$.

Proof. It follows immediately from the fact that $r_2(\chi, w_0(M)) = \{\alpha \in \Pi : w_0(M)\alpha \in \Delta^+\} = \Pi \setminus \Pi_M$. \square

Since $W(\chi) = \{1\}$, the bilinear form in this case is $(w_1 : w_2) = (-v)^{\tau(w_1, w_2)}$, for every $w_1, w_2 \in W$. The radical can also be easily described. For every $\Pi_M \subset \Pi$, define

$$S_M = \{w \in W : w\alpha \notin \Delta^+, \forall \alpha \in \Pi_M \text{ and } w\beta \in \Delta^+, \forall \beta \in \Pi \setminus \Pi_M\}. \tag{4.1.3}$$

Note that $w_0(M) \in S_M$. Then it follows immediately that a basis for *Rad* is

$$\bigsqcup_{\Pi_M \subset \Pi} \{w_0(M) - w' : w' \in S_P, w' \neq w_0(M)\}. \tag{4.1.4}$$

(Note that $\dim \mathcal{K}(\chi) = |W|$, while $\dim \mathcal{K}(\chi)/\text{Rad} = 2^{|\Pi|}$.)

Definition 3. For every subset $\Pi_{M'} \subset \Pi$, define $\xi_{M'} = \sum_{\Pi_M \subset \Pi_{M'}} v^{|\Pi_{M'}| - |\Pi_M|} w_0(M)$.

Proposition 6. If $\Pi_{M'} \subset \Pi$, we have the following identities:

- (1) $(\xi_{M'}, \xi_{M'}) = \sum_{\Pi_M \subset \Pi_{M'}} (-v^2)^{|\Pi_{M'}| - |\Pi_M|} = \sum_{\Pi_M \subset \Pi_{M'}} (-v^2)^{|\Pi_M|}$.
(2) $(w_0 : \xi_{M'}) = (-v)^{|\Pi| - |\Pi_{M'}|} \cdot (\xi_{M'} : \xi_{M'})$.

Proof. To prove (1), it suffices to prove the identity when $M' = G$. We will show first that $(\xi_G : w_0(M_1)) = 0$, for all $\Pi_{M_1} \subsetneq \Pi$. We have $(\xi_G : w_0(M_1)) = \sum_{\Pi_M \subset \Pi} v^{|\Pi| - |\Pi_M|} (-v)^{|\Pi_M \vee \Pi_{M_1}|} = (-1)^{|\Pi_{M_1}|} v^{|\Pi| + |\Pi_{M_1}|} \sum_{\Pi_M \subset \Pi} x_{M, M_1}$, where we denoted $x_{M, M_1} = (-1)^{|\Pi_M|} (v^{-2})^{|\Pi_M \cap \Pi_{M_1}|}$.

Let α be a root such that $\alpha \in \Pi \setminus \Pi_{M_1}$. Then, for every $\Pi_M \subset \Pi \setminus \{\alpha\}$, $x_{M, M_1} = -x_{M \cup \{\alpha\}, M_1}$. The last sum can be written as $\sum_{\Pi_M \subset \Pi \setminus \{\alpha\}} x_{M, M_1} +$

$$\sum_{\Pi_M \subset \Pi \setminus \{\alpha\}} x_{M \cup \{\alpha\}, M_1} = 0.$$

This implies that $(\xi_G : \xi_G) = (\xi_G : w_0) = \sum_{\Pi_M \subset \Pi_{M'}} (-v^2)^{|\Pi_{M'}| - |\Pi_M|}$.

Formula (2) follows immediately from (1):

$$(w_0 : \xi_{M'}) = \sum_{\Pi_M \subset \Pi_{M'}} v^{|\Pi_{M'}| - |\Pi_M|} \cdot (-v)^{|\Pi| - |\Pi_M|} = (-v)^{|\Pi| - |\Pi_{M'}|} (\xi_{M'} : \xi_{M'}).$$

□

The basis elements in \mathcal{Z}_+ , respectively \mathcal{U}_+ are obtained from those of \mathcal{Z}_- , respectively \mathcal{U}_- by multiplication by w_0 on the right. From proposition 6 and in view of the algorithm, we can determine the basis elements of \mathcal{Z}_- and \mathcal{Z}_+ .

Corollary 3. The bases are $\mathcal{Z}_- = \{\xi_{M'} : \Pi'_M \subset \Pi\}$ and $\mathcal{U}_- = \{w_0(M') : \Pi_{M'} \subset \Pi\}$. Moreover, after the sign normalization, the polynomials are

$$c_{M_1, M_2} = \begin{cases} 0, & \text{if } \Pi_{M_1} \not\subseteq \Pi_{M_2}, \\ v^{|\Pi_{M_2}| - |\Pi_{M_1}|}, & \text{if } \Pi_{M_1} \subseteq \Pi_{M_2}. \end{cases} \quad (4.1.5)$$

Proof. It remains to verify that $\xi_G = w_0 - \sum_{\Pi_{M'} \subsetneq \Pi} (-v)^{|\Pi| - |\Pi_{M'}|} \xi_{M'}$. The right hand side equals $RHS = w_0 - \sum_{\Pi_{M'} \subsetneq \Pi} \sum_{\Pi_M \subset \Pi_{M'}} \Pi_M (-1)^{|\Pi| - |\Pi_{M'}|} v^{|\Pi| - |\Pi_M|} w_0(M)$.

We rewrite it as $RHS = w_0 - \sum_{\Pi_M \subsetneq \Pi} v^{|\Pi| - |\Pi_M|} w_0(M) \left(-1 + \sum_{\Pi_{M'} \supset \Pi_M} (-1)^{|\Pi| - |\Pi_{M'}|} \right)$.

Finally, $\sum_{\Pi_{M'} \supset \Pi_M} (-1)^{|\Pi| - |\Pi_{M'}|} = (-1)^{|\Pi| - |\Pi_M|} \sum_{S \subset \Pi \setminus \Pi_M} (-1)^{|S|} = 0$.

□

In terms of Kazhdan-Lusztig polynomials, this result is formulated as follows:

$$P_{M_1, M_2}(q) = \begin{cases} 0, & \text{if } \Pi_{M_1} \not\subseteq \Pi_{M_2}, \\ 1, & \text{if } \Pi_{M_1} \subseteq \Pi_{M_2}. \end{cases} \quad (4.1.6)$$

4.2 Zelevinsky's example in $gl(4)$

This is one of the first examples of nontrivial Kazhdan-Lusztig polynomials (see 11.4 in [19]). Consider $\chi = (2, 0, 0, -2)$ in $gl(4)$ for simple roots $\Pi = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \epsilon_3 - \epsilon_4\}$. All local systems are trivial. The list of orbits is:

Dimension	s	G -saturation	\mathcal{Z}_-	\mathcal{U}_-
0	(0, 0, 0, 0)	(1 ⁴)	$\frac{v}{1+v^2}[1]$	$\frac{v}{1+v^2}[1]$
2 _a	(1, -1, 0, 0)	(211)	$[s_1] + v[1]$	$[s_1] + \frac{v}{1+v^2}[1]$
2 _b	(0, 0, 1, -1)	(211)	$[s_3] + v[1]$	$[s_3] + \frac{v}{1+v^2}[1]$
3	(1, -1, 1, -1)	(22)	$[s_1 s_3] + v[s_1] + v[s_3] + v^2[1]$	$[s_1 s_3]$
4	(2, 0, 0, -2)	(31)	$\frac{v}{1+v^2}[w_0] - v[s_1 s_3] + \frac{v^3}{1+v^2}[1]$	$\frac{v}{1+v^2}[w_0]$

The change of basis matrix from \mathcal{Z}_- to \mathcal{U}_- is:

$$\begin{pmatrix} 1 & -v^2 & -v^2 & v + v^3 & v^4 \\ 0 & 1 & 0 & -v & -v^2 \\ 0 & 0 & 1 & -v & -v^2 \\ 0 & 0 & 0 & 1 & v \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and the matrix of Kazhdan-Lusztig polynomials is:

$$\left(\begin{array}{c|ccccc} & 0 & 2_a & 2_b & 3 & 4 \\ \hline 0 & 1 & 1 & 1 & 1+q & 1 \\ 2_a & 0 & 1 & 0 & 1 & 1 \\ 2_b & 0 & 0 & 1 & 1 & 1 \\ 3 & 0 & 0 & 0 & 1 & 1 \\ \hline 4 & 0 & 0 & 0 & 0 & 1 \end{array} \right).$$

The action of the involution IM is:

$$IM(5) = 0, \quad IM(3) = 3, \quad IM(2_a) = 2_b. \quad (4.2.1)$$

In the following examples, in subsections 4.3-4.5, the multiplicities were previously computed in [15].

4.3 $\chi = (1, 1)$ in $sp(4)$

Consider $\chi = (1, 1)$, the middle element of the nilpotent orbit (22) in $sp(4)$. There are 3 orbits in $Orb_2(\chi)$, the open orbit with two local systems. The first one listed below is the trivial. Each Weyl group coset $W/W(\chi)$ is given by the action of a representative element on χ .

Dimension	s	\mathcal{Z}_-	\mathcal{U}_-
0	(0, 0)	$\frac{v}{1+v^2}[1, 1]$	$\frac{v}{1+v^2}[1, 1]$
2	(0, 1)	$[1, -1] + v[1, 1]$	$[1, -1] + \frac{v}{1+v^2}[1, 1]$
3	(1, 1)	$\frac{v}{1+v^2}[-1, -1] - v[1, -1] - \frac{v^2}{1+v^2}[1, 1]$	$\frac{v}{1+v^2}[-1, -1]$
		$[-1, 1] + \frac{v}{1+v^2}[-1, -1] - \frac{v^2}{1+v^2}[1, 1]$	$[-1, 1] + \frac{v}{1+v^2}[-1, -1]$

The change of basis matrix is $\left(\begin{array}{cc|cc} 1 & -v^2 & -v^3 & v \\ 0 & 1 & v & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)$, and the matrix of

Kazhdan-Lusztig polynomials is

$$\left(\begin{array}{c|cc|cc} & 0 & 2 & \mathfrak{3}_{triv} & \mathfrak{3}_{sgn} \\ \hline 0 & 1 & 1 & 1 & q \\ 2 & 0 & 1 & 1 & 0 \\ \hline \mathfrak{3}_{triv} & 0 & 0 & 1 & 0 \\ \mathfrak{3}_{sgn} & 0 & 0 & 0 & 1 \end{array} \right).$$

The action of the involution IM is

$$IM(3) = 1, \quad IM(4) = 2. \quad (4.3.1)$$

4.4 $\chi = (3, 1, 1)$ in $sp(6)$

The central character is $\chi = (3, 1, 1)$, the middle element of the triangular nilpotent $(4, 2)$ in $sp(6)$. There are 8 orbits in $Orb_2(\chi)$, two orbits (one of which is the open orbit) with two local systems. We list the parameterization of these orbits, the dimensions, the corresponding Levi subalgebras and the basis elements \mathcal{Z}_- and \mathcal{U}_- . The bases \mathcal{Z}_+ and \mathcal{U}_+ are obtained by multiplication by w_0 .

We encode the cosets $W/W(\chi)$ by the W action on $(3, 1, 1)$.

Dim	s	\mathcal{Z}_-
0	(0, 0, 0)	$\frac{1}{v+v-\Gamma}[3, 1, 1]$
2_a	(1, -1, 0)	$[1, 3, 1] + v[3, 1, 1]$
2_b	(0, 0, 1)	$[3, 1, -1] + v[3, 1, 1]$
3_a	(1, -1, 1)	$[1, 3, -1] + v[1, 3, 1] + v[3, 1, -1] + v^2[3, 1, 1]$
$3_{b,t}$	(0, 1, 1)	$\frac{1}{v+v-\Gamma}[3, -1, -1] - v[3, 1, -1] - \frac{v}{v+v-\Gamma}[3, 1, 1]$
$3_{b,s}$		$[3, -1, 1] + \frac{1}{v+v-\Gamma}[3, -1, -1] - \frac{v}{v+v-\Gamma}[3, 1, 1]$
4_a	(2, 0, 2)	$[-1, 1, 3] + v[1, 3, -1] + v[3, -1, 1] + v^2[3, 1, -1]$
4_b	(3, 1, 0)	$[1, -3, -1] + v[1, 1, 3] + v[1, 3, -1] + v^2[1, 3, 1]$
5_t	(3, 1, 1)	$\frac{1}{v+v-\Gamma}[-3, -1, -1] - v[-1, 1, 3] - v[1, -3, -1]$ $-\frac{v^2}{v+v-\Gamma}[3, -1, -1] - v^2[1, 1, 3] - v^2[1, 3, -1]$ $-\frac{v^2}{v+v-\Gamma}[3, -1, 1] - v^3[1, 3, 1]$
5_s		$[-3, -1, 1] + \frac{1}{v+v-\Gamma}[-3, -1, -1] + v[-1, 1, 3] + v[1, 3, 1]$ $-\frac{v^2}{v+v-\Gamma}[3, -1, -1] + v^2[1, 3, -1] + v^2[3, 1, 1] + v^3[3, 1, -1]$

s	\mathcal{U}_-	G -saturation
(0, 0, 0)	$\frac{1}{v+v-\Gamma}[3, 1, 1]$	(1^6)
(1, -1, 0)	$[1, 3, 1] + \frac{1}{v+v-\Gamma}[3, 1, 1]$	(221^2)
(0, 0, 1)	$[3, 1, -1] + \frac{1}{v+v-\Gamma}[3, 1, 1]$	(21^4)
(1, -1, 1)	$[1, 3, -1]$	$(2, 2, 2)$
(0, 1, 1)	$\frac{1}{v+v-\Gamma}[3, -1, -1]$	(221^2)
	$[3, -1, 1] + \frac{1}{v+v-\Gamma}[3, -1, -1]$	
(2, 0, 2)	$[-1, 1, 3]$	(33)
(3, 1, 0)	$[1, -3, -1] + v[1, 1, 3] + v^2[1, 3, -1] + \frac{v^2}{v+v-\Gamma}[1, 3, 1]$	(41^2)
(3, 1, 1)	$\frac{1}{v+v-\Gamma}[-3, -1, -1]$	(42)
	$[-3, -1, 1] + \frac{1}{v+v-\Gamma}[-3, -1, -1]$	

The change of basis matrix from \mathcal{Z}_- to \mathcal{U}_- is:

$$\left(\begin{array}{cccccccc|cc} 1 & -v^2 & -v^2 & (v+v^3) & -v^3 & v & -(v^2+v^4) & -v^4 & -v^5 & v^3 \\ 0 & 1 & 0 & -v & 0 & 0 & v^2 & v^2 & v^3 & -v \\ 0 & 0 & 1 & -v & v & 0 & v^2 & v^2 & v^3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -v & -v & -v^2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & v & 0 & v^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -v & 0 & 0 & v^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -v \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & v & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right),$$

and the matrix of Kazhdan-Lusztig polynomials is

$$\left(\begin{array}{c|cccccccc|cc} & 0 & 2_a & 2_b & 3_a & 3_{b,t} & 3_{b,s} & 4_a & 4_b & 5_t & 5_s \\ \hline 0 & 1 & 1 & 1 & 1+q & 1 & q & 1+q & q & 1 & q \\ 2_a & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & q \\ 2_b & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 3_a & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 3_{b,t} & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 3_{b,s} & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 4_a & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 4_b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ \hline 5_t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 5_s & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right).$$

The action of the involution IM is:

$$IM(1) = 9, \quad IM(2) = 5, \quad IM(3) = 10, \quad IM(4) = 7, \quad IM(6) = 8. \quad (4.4.1)$$

4.5 G_2

There are five nilpotent orbits. The regular orbit is a particular case of section 4.1, and we will also ignore the trivial orbit.

4.5.1 $G_2(a_1)$

The central character is χ , the middle element of the nilpotent $G_2(a_1)$. There are 4 orbits of $G(\chi)$ on $\mathfrak{g}_2(\chi)$, and it turns out they are distinguished by their G -saturation. They are:

Dimension	G -saturation
0	0
2	A_1
3	\tilde{A}_1
4	$G_2(a_1)$.

The closure ordering is

$$0 - 2 - 3 - 4.$$

The stabilizer of a point in the dense orbit (4-dimensional) is S_3 , but only 2 local systems appear for the equal parameter case (the extra local system is cuspidal).

The matrix of Kazhdan-Lusztig polynomials is:

$$\left(\begin{array}{c|ccc|cc} & 0 & 2 & 3 & 4_t & 4_s \\ \hline 0 & 1 & 1 & q+1 & 1 & q \\ 2 & 0 & 1 & 1 & 1 & 0 \\ 3 & 0 & 0 & 1 & 1 & 1 \\ \hline 4_t & 0 & 0 & 0 & 1 & 0 \\ 4_s & 0 & 0 & 0 & 0 & 1 \end{array} \right)$$

In terms of the classical Langlands classification for the p -adic group \mathcal{G} of type G_2 whose dual is G , the rows correspond to the induced standard modules from: the Borel subgroup, the parabolic of type A_1 short, the parabolic of type A_1 long, and two discrete series (the first generic) respectively. We denote these induced modules by $X(0)$ (this is the full unramified principal series), $X(A_1^s)$, $X(A_1^l)$, and $DS(g)$, $DS(ng)$ respectively. The columns correspond to the Langlands quotients: $\overline{X}(0)$, $\overline{X}(A_1^s)$, $\overline{X}(A_1^l)$, and $DS(g)$, $DS(ng)$ respectively. Therefore the character decompositions are:

$$X(0) = \overline{X}(0) + \overline{X}(A_1^s) + 2 \cdot \overline{X}(A_1^l) + DS(g) + DS(ng); \quad (4.5.1)$$

$$X(A_1^s) = \overline{X}(A_1^s) + \overline{X}(A_1^l) + DS(g); \quad (4.5.2)$$

$$X(A_1^l) = \overline{X}(A_1^l) + DS(g) + DS(ng); \quad (4.5.3)$$

$$DS(g) = DS(g); \quad (4.5.4)$$

$$DS(ng) = DS(ng). \quad (4.5.5)$$

Finally,

$$IM(DS(g)) = \overline{X}(0), \quad IM(DS(ng)) = \overline{X}(A_1^s), \quad IM(\overline{X}(A_1^l)) = \overline{X}(A_1^l). \quad (4.5.6)$$

4.5.2 A_1 or \tilde{A}_1

If χ is the middle element of the nilpotent A_1 or \tilde{A}_1 , then $Orb_2(\chi)$ has only two elements, the zero orbit and the dense orbit of dimension one, and the component groups are trivial. The matrix of Kazhdan-Lusztig polynomials is, in both cases,

$$\left(\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1 & 1 \\ 1 & 0 & 1 \end{array} \right).$$

5 Polynomials for F_4

There are 16 nilpotent orbits in F_4 . We compute the polynomials for the central characters χ that are middle elements of nilpotent orbits. The most interesting example is when χ is the middle element of the nilpotent $F_4(a_3)$. This is the one

with component group of S_4 . We first present this example in detail and then record the other 15 cases.

We also give the Iwahori-Matsumoto dual of the tempered modules. When we write $IM(\xi) = \xi'$, we mean that the IM dual of the simple module parameterized by ξ is the simple module parameterized by ξ' .

The simple roots we use for F_4 are $\alpha_1 = (1, -1, -1, -1)$, $\alpha_2 = (0, 0, 0, 2)$, $\alpha_3 = (0, 0, 1, -1)$, $\alpha_4 = (0, 1, -1, 0)$. In these coordinates, the “most interesting” χ is $(3, 1, 1, 1)$.

The notation for orbits and local systems is as explained at the beginning of section 4. The only change is that for the open orbit at $\chi = (3, 1, 1, 1)$, the component group being S_4 , we use a subscript denoting the partition of 4 which labels the corresponding irreducible representation of S_4 .

5.1 $\chi = (3, 1, 1, 1)$ in F_4

This is the middle element of the nilpotent orbit $F_4(a_3)$. There are 12 orbits and a total of 20 local systems. (This was previously known by [5]). The component group of the stabilizer of a point in the open orbit is S_4 , so there are irreducible 5 local systems, one of which is cuspidal in the sense of Lusztig. We will not consider it because it doesn't parameterize $mod_\chi(\mathbb{H})$. Therefore, our matrix has 19 columns and rows (corresponding to 12 orbits), where the last 4 correspond to the open orbit.

The list of orbits follows. For each orbit, we give a label which encodes the dimension as well, the semisimple element s (from which the good parabolic is constructed), the G -saturation of the orbit, and the component group of the stabilizer in $G(\chi)$ of a point in the orbit.

Dimension	s	G -saturation	Components
0	$(0, 0, 0, 0)$	0	1
4	$(0, 0, 0, 1)$	A_1	1
6	$(0, 0, 1, 1)$	\tilde{A}_1	$\mathbb{Z}/2\mathbb{Z}$
7'	$(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2})$	$A_1 + \tilde{A}_1$	1
7''	$(0, 1, 1, 1)$	$A_1 + \tilde{A}_1$	1
8'	$(1, -1, 1, 1)$	A_2	$\mathbb{Z}/2\mathbb{Z}$
8''	$(2, 0, 0, 2)$	\tilde{A}_2	1
9	$(1, 0, 1, 2)$	$\tilde{A}_1 + A_2$	1
10'	$(2, 0, 1, 2)$	$A_1 + \tilde{A}_2$	1
10''	$(2, 1, 1, 2)$	B_2	$\mathbb{Z}/2\mathbb{Z}$
11	$(\frac{5}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2})$	$C_3(a_1)$	$\mathbb{Z}/2\mathbb{Z}$
12	$(3, 1, 1, 1)$	$F_4(a_3)$	S_4

The Kazhdan-Lusztig polynomials are in the following matrix. There are 19 columns, each corresponding to one of the local systems in the table above. Due to the size of the matrix of polynomials, we break it into three parts. There is a subtle issue of identifying the three nontrivial local systems on the open orbit 12. Note in the matrix, that they are distinguished by their multiplicity in the first row. In representation theoretic language, this means that the three

corresponding nongeneric discrete series are distinguished by their multiplicity in the spherical principal series. Then to complete the identification, we referred to [4].

Columns 1-8:

	0	4	6_t	6_s	7_a	7_b	$8_{a,t}$	$8_{a,s}$
0	1	$1+q$	$1+q+q^2$	$q+q^2$	$1+2q+2q^2+q^3$	$1+q$	$1+q+q^2$	q
4	0	1	1	0	$1+q$	1	$1+q$	q
6_t	0	0	1	0	1	1	1	0
6_s	0	0	0	1	1	0	0	0
7_a	0	0	0	0	1	0	1	0
7_b	0	0	0	0	0	1	0	0
$8_{a,t}$	0	0	0	0	0	0	1	0
$8_{a,s}$	0	0	0	0	0	0	0	1
8_b	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0
10_a	0	0	0	0	0	0	0	0
$10_{b,t}$	0	0	0	0	0	0	0	0
$10_{b,s}$	0	0	0	0	0	0	0	0
11_t	0	0	0	0	0	0	0	0
11_s	0	0	0	0	0	0	0	0
$12_{(4)}$	0	0	0	0	0	0	0	0
$12_{(31)}$	0	0	0	0	0	0	0	0
$12_{(22)}$	0	0	0	0	0	0	0	0
$12_{(211)}$	0	0	0	0	0	0	0	0

Columns 9-13:

	8_b	9	10_a	$10_{b,t}$	$10_{b,s}$
0	$1+q+q^2$	$1+q+q^2+q^3$	$1+q+2q^2+q^3+q^4$	$1+q$	$q+2q^2+q^3$
4	1	$1+q+q^2$	$1+q+q^2$	$1+q$	$q+q^2$
6_t	1	1	$1+q$	1	q
6_s	1	0	q	0	q
7_a	1	1	$1+q$	1	q
7_b	0	1	$1+q$	1	q
$8_{a,t}$	0	1	1	1	0
$8_{a,s}$	0	1	0	1	0
8_b	1	0	1	0	0
9	0	1	1	1	0
10_a	0	0	1	0	0
$10_{b,t}$	0	0	0	1	0
$10_{b,s}$	0	0	0	0	1
11_t	0	0	0	0	0
11_s	0	0	0	0	0
$12_{(4)}$	0	0	0	0	0
$12_{(31)}$	0	0	0	0	0
$12_{(22)}$	0	0	0	0	0
$12_{(211)}$	0	0	0	0	0

Columns 14-19:

	11_t	11_s	$12_{(4)}$	$12_{(31)}$	$12_{(22)}$	$12_{(211)}$
0	$1+2q+2q^2+q^3$	$q+q^2+q^3+q^4$	1	$q+q^2$	$q+q^2+q^3$	q^3
4	$1+2q+q^2$	q	1	q	$q+q^2$	0
6_t	$1+2q$	q	1	q	$q+q^2$	0
6_s	q	$q+q^2$	0	q	0	q^2
7_a	$1+2q$	q	1	q	q	0
7_b	$1+q$	q	1	q	q^2	0
$8_{a,t}$	$1+q$	0	1	0	q	0
$8_{a,s}$	0	0	0	0	0	0
8_b	$1+q$	$1+q$	1	$1+q$	q	q
9	1	0	1	0	0	0
10_a	1	1	1	1	0	0
$10_{b,t}$	1	0	1	0	1	0
$10_{b,s}$	1	0	0	1	0	0
11_t	1	0	1	1	1	0
11_s	0	1	0	1	0	1
$12_{(4)}$	0	0	1	0	0	0
$12_{(31)}$	0	0	0	1	0	0
$12_{(22)}$	0	0	0	0	1	0
$12_{(211)}$	0	0	0	0	0	1

The Iwahori-Matsumoto involution gives:

$$IM(12_{(4)}) = 0, \quad IM(12_{(31)}) = 4, \quad IM(12_{(22)}) = 6_t, \quad IM(12_{(211)}) = 8_{a,s}. \quad (5.1.1)$$

From the list of polynomials, we find that the closure ordering in $Orb_2(\chi)$ is as in figure 5.1.

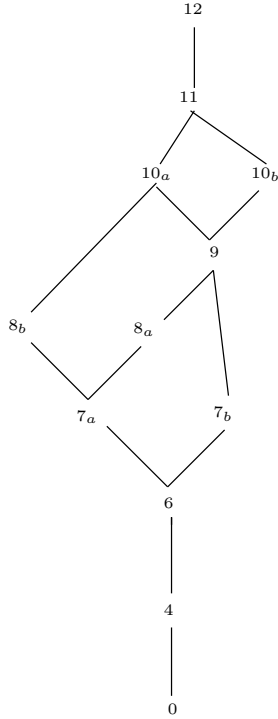


Figure 1: The closure ordering for $Orb_2(\chi)$ in F_4 , where $\chi = (3, 1, 1, 1)$.

5.2 The other 15 cases

For each χ we give the list of orbits and local systems, and the matrix of polynomials.

5.2.1 $\chi = (11, 5, 3, 1)$, $\mathcal{O} = F_4$

This is a particular case of the general case $\chi = 2\hat{\rho}$ in section 4.

5.2.2 $\chi = (7, 3, 1, 1)$, $\mathcal{O} = F_4(a_1)$

Dimension	s	G -saturation	Components
0	(0, 0, 0, 0)	0	1
1	$(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$	A_1	1
2_a	(0, 0, 0, 1)	A_1	1
2_b	(0, 1, -1, 0)	\tilde{A}_1	1
3_a	(1, -1, -1, 1)	A_2	1
3_b	$(\frac{1}{2}, \frac{1}{2}, -\frac{3}{2}, -\frac{1}{2})$	$A_1 + \tilde{A}_1$	1
3_c	(0, 1, -1, 1)	$A_1 + \tilde{A}_1$	1
3_d	(0, 0, 1, 1)	\tilde{A}_1	$\mathbb{Z}/2\mathbb{Z}$
4_a	(0, 2, 0, 2)	\tilde{A}_2	1
4_b	(0, 3, 0, 1)	C_2	1
4_c	(1, 0, -2, 1)	$A_2 + \tilde{A}_1$	1
4_d	(2, -2, 1, 1)	B_2	$\mathbb{Z}/2\mathbb{Z}$
5_a	(3, 3, -3, 1)	B_3	1
5_b	(0, 3, 1, 1)	$C_3(a_1)$	$\mathbb{Z}/2\mathbb{Z}$
5_c	$(\frac{3}{2}, \frac{1}{2}, -\frac{3}{2}, -\frac{7}{2})$	C_3	1
6	$(7, 3, 1, 1)$	$F_4(a_1)$	$\mathbb{Z}/2\mathbb{Z}$

$$IM(6_t) = 0, \quad IM(6_s) = 2_a. \quad (5.2.1)$$

Columns 1 - 14:

	0	1	2_a	2_b	3_a	3_b	3_c	$3_{d,t}$	$3_{d,s}$	4_a	4_b	4_c	$4_{d,t}$	$4_{d,s}$
0	1	1	1	1	1	1	$1+q$	1	q	$1+q$	1	$1+q$	1	q
1	0	1	0	0	1	1	0	0	0	0	0	$1+q$	1	q
2_a	0	0	1	0	1	0	1	0	1	1	1	1	1	0
2_b	0	0	0	1	0	1	1	0	0	1	1	1	0	0
3_a	0	0	0	0	1	0	0	0	0	0	0	1	1	0
3_b	0	0	0	0	0	1	0	0	0	0	0	1	0	0
3_c	0	0	0	0	0	0	1	0	0	1	1	1	0	0
$3_{d,t}$	0	0	0	0	0	0	0	1	0	1	0	0	1	0
$3_{d,s}$	0	0	0	0	0	0	0	0	1	1	0	0	0	1
4_a	0	0	0	0	0	0	0	0	0	1	0	0	0	0
4_b	0	0	0	0	0	0	0	0	0	0	1	0	0	0
4_c	0	0	0	0	0	0	0	0	0	0	0	1	0	0
$4_{d,t}$	0	0	0	0	0	0	0	0	0	0	0	0	1	0
$4_{d,s}$	0	0	0	0	0	0	0	0	0	0	0	0	0	1
5_a	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$5_{b,t}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$5_{b,s}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5_c	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6_t	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6_s	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Columns 15 - 20:

	5_a	$5_{b,t}$	$5_{b,s}$	5_c	6_t	6_s
0	1	1	q	$1+q$	1	q
1	1	0	0	$1+q$	1	q
2_a	1	1	0	1	1	0
2_b	1	1	q	1	1	q
3_a	1	0	0	1	1	0
3_b	1	0	0	1	1	q
3_c	1	1	0	1	1	0
$3_{d,t}$	0	1	0	1	1	0
$3_{d,s}$	0	0	1	1	0	1
4_a	0	1	1	1	1	1
4_b	1	1	0	0	1	0
4_c	1	0	0	1	1	0
$4_{d,t}$	0	0	0	1	1	0
$4_{d,s}$	0	0	0	1	0	1
5_a	1	0	0	0	1	0
$5_{b,t}$	0	1	0	0	1	0
$5_{b,s}$	0	0	1	0	0	1
5_c	0	0	0	1	1	1
6_t	0	0	0	0	1	0
6_s	0	0	0	0	0	1

5.2.3 $\chi = (5, 3, 1, 1)$, $\mathcal{O} = F_4(a_2)$

Dimension	s	G -saturation	Components
0	(0, 0, 0, 0)	0	1
2	(0, 1, -1, 0)	\bar{A}_1	1
3	(0, 0, 0, 1)	A_1	1
4_a	(0, 1, -1, 1)	$A_1 + \bar{A}_1$	1
4_b	(0, 0, 1, 1)	\bar{A}_1	$\mathbb{Z}/2\mathbb{Z}$
5_a	(0, 2, 0, 2)	\bar{A}_2	1
5_b	$(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2})$	$A_1 + \bar{A}_1$	1
5_c	(0, 3, 0, 1)	C_2	1
6_a	$(\frac{1}{2}, \frac{3}{2}, -\frac{1}{2}, \frac{5}{2})$	$A_1 + \bar{A}_2$	1
6_b	(1, -1, 1, 1)	A_2	$\mathbb{Z}/2\mathbb{Z}$
6_c	(0, 3, 1, 1)	$C_3(a_1)$	$\mathbb{Z}/2\mathbb{Z}$
7_a	(3, 3, 1, 3)	B_3	1
7_b	$(\frac{3}{2}, \frac{7}{2}, \frac{1}{2}, \frac{5}{2})$	C_3	1
8	(5, 3, 1, 1)	$F_4(a_2)$	$\mathbb{Z}/2\mathbb{Z}$

$$IM(8_t) = 0, \quad IM(8_s) = 4_{b,s}. \quad (5.2.2)$$

Columns 1 - 9		0	2	3	4_a	$4_{b,t}$	$4_{b,s}$	5_a	5_b	5_c
0		1	1	$1+q$	$1+q$	$1+q$	q	$1+2q+q^2$	$1+q$	$1+q$
2		0	1	0	1	0	0	$1+q$	0	$1+q$
3		0	0	1	1	1	0	1	1	1
4_a		0	0	0	1	0	0	1	0	1
$4_{b,t}$		0	0	0	0	1	0	1	1	0
$4_{b,s}$		0	0	0	0	0	1	1	1	0
5_a		0	0	0	0	0	0	1	0	0
5_b		0	0	0	0	0	0	0	1	0
5_c		0	0	0	0	0	0	0	0	1
6_a		0	0	0	0	0	0	0	0	0
$6_{b,t}$		0	0	0	0	0	0	0	0	0
$6_{b,s}$		0	0	0	0	0	0	0	0	0
$6_{c,t}$		0	0	0	0	0	0	0	0	0
$6_{c,s}$		0	0	0	0	0	0	0	0	0
7_a		0	0	0	0	0	0	0	0	0
7_b		0	0	0	0	0	0	0	0	0
8_t		0	0	0	0	0	0	0	0	0
8_s		0	0	0	0	0	0	0	0	0

Columns 10 - 18:		6_a	$6_{b,t}$	$6_{b,s}$	$6_{c,t}$	$6_{c,s}$	7_a	7_b	8_t	8_s
0		$1+q$	1	q	$1+q$	q	$1+2q+q^2$	$1+q$	1	q
2		1	0	0	$1+q$	q	$1+q$	$1+q$	1	q
3		1	1	q	1	0	$1+2q$	1	1	q
4_a		1	0	0	1	0	$1+q$	1	1	q
$4_{b,t}$		1	1	0	1	0	$1+q$	1	1	0
$4_{b,s}$		1	0	0	0	1	0	1	0	0
5_a		1	0	0	1	1	1	2	1	0
5_b		1	1	0	0	0	1	1	1	0
5_c		0	0	0	1	0	1	1	1	q
6_a		1	0	0	0	0	1	1	1	0
$6_{b,t}$		0	1	0	0	0	1	0	1	0
$6_{b,s}$		0	0	1	0	0	1	0	0	1
$6_{c,t}$		0	0	0	1	0	1	1	1	0
$6_{c,s}$		0	0	0	0	1	0	1	0	0
7_a		0	0	0	0	0	1	0	1	1
7_b		0	0	0	0	0	0	1	1	0
8_t		0	0	0	0	0	0	0	1	0
8_s		0	0	0	0	0	0	0	0	1

5.2.4 $\chi = (5, 1, 1, 1)$, $\mathcal{O} = B_3$

Dim.	s	G -sat.	Comp.	0	1	3	4	5_t	5_s	$6_{a,t}$	$6_{a,s}$	6_b	7
0	(0, 0, 0, 0)	0	1	1	1	1	1	$1+q^2$	q	$1+q^2$	q	1	1
1	$(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$	A_1	1	0	1	0	1	0	0	$1+q^2$	q	0	1
3	(0, 0, 0, 1)	A_1	1	0	0	1	1	1	0	1	0	1	1
4	(1, -1, -1, 1)	A_2	1	0	0	0	1	0	0	1	0	0	1
5	(0, 0, 1, 1)	\bar{A}_1	$\mathbb{Z}/2\mathbb{Z}$	0	0	0	1	0	0	1	0	0	1
6_a	(2, -2, 1, 1)	B_2	$\mathbb{Z}/2\mathbb{Z}$	0	0	0	0	1	0	1	0	0	1
6_b	(0, 1, 1, 1)	$A_1 + \bar{A}_1$	1	0	0	0	0	0	0	0	1	0	0
7	(5, 1, 1, 1)	B_3	1	0	0	0	0	0	0	0	0	1	1

$$IM(7) = 0. \quad (5.2.3)$$

5.2.5 $\chi = (5, 3, 1, 0)$, $\mathcal{O} = C_3$

Dim.	s	G -sat.	Comp.
0	(0, 0, 0, 0)	0	1
1 _a	(1, -1, 0, 0)	\tilde{A}_1	1
1 _b	(0, 1, -1, 0)	\tilde{A}_1	1
1 _c	(0, 0, 1, 0)	\tilde{A}_1	1
2 _a	(2, 0, -2, 0)	\tilde{A}_2	1
2 _b	(1, -1, 1, 0)	$A_1 + \tilde{A}_1$	1
2 _c	(0, 3, 1, 0)	B_2	1
3	(5, 3, 1, 0)	C_3	1

	0	1 _a	1 _b	1 _c	2 _a	2 _b	2 _c	3
0	1	1	1	1	1	1	1	1
1 _a	0	1	0	0	1	1	0	1
1 _b	0	0	1	0	1	0	1	1
1 _c	0	0	0	1	0	1	1	1
2 _a	0	0	0	0	1	0	0	1
2 _b	0	0	0	0	0	1	0	1
2 _c	0	0	0	0	0	0	1	1
3	0	0	0	0	0	0	0	1

$$IM(3) = 0. \quad (5.2.4)$$

5.2.6 $\chi = (3, 1, 1, 0)$, $\mathcal{O} = C_3(a_1)$

Dim.	s	G -sat.	Comp.
0	(0, 0, 0, 0)	0	1
2 _a	(1, -1, 0, 0)	\tilde{A}_1	1
2 _b	(0, 0, 1, 0)	\tilde{A}_1	1
3 _a	(1, -1, 1, 0)	$A_1 + \tilde{A}_1$	1
3 _b	(0, 1, 1, 0)	\tilde{A}_1	$\mathbb{Z}/2\mathbb{Z}$
4 _a	(2, 0, 2, 0)	\tilde{A}_2	1
4 _b	(3, 0, 1, 0)	B_2	1
5	(3, 1, 1, 0)	$C_3(a_1)$	$\mathbb{Z}/2\mathbb{Z}$

	0	2 _a	2 _b	3 _a	3 _{b,t}	3 _{b,s}	4 _a	4 _b	5 _t	5 _s
0	1	1	1	1+q	1	q	1+q	1	1	q
2 _a	0	1	0	1	0	0	1	1	1	q
2 _b	0	0	1	1	1	0	1	1	1	0
3 _a	0	0	0	1	0	0	1	1	1	0
3 _{b,t}	0	0	0	0	1	0	1	0	1	0
3 _{b,s}	0	0	0	0	0	1	1	0	0	1
4 _a	0	0	0	0	0	0	1	0	1	1
4 _b	0	0	0	0	0	0	0	1	1	0
5 _t	0	0	0	0	0	0	0	0	1	0
5 _s	0	0	0	0	0	0	0	0	0	1

$$IM(5_t) = 0, \quad IM(5_s) = 2_b. \quad (5.2.5)$$

5.2.7 $\chi = (\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2})$, $\mathcal{O} = A_1 + \tilde{A}_2$

Dim.	s	G -sat.	Comp.
0	(0, 0, 0, 0)	0	1
1	($\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$)	A_1	1
3	(0, 1, 0, 1)	\tilde{A}_1	1
4 _a	($\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}$)	$A_1 + \tilde{A}_1$	1
4 _b	(2, 2, 0, 0)	\tilde{A}_2	1
5	($\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}$)	$A_1 + \tilde{A}_2$	1

	0	1	3	4 _a	4 _b	5
0	1	1	1+q	1+q	1	1
1	0	1	0	1+q	0	1
3	0	0	1	1	1	1
4 _a	0	0	0	1	0	1
4 _b	0	0	0	0	1	1
5	0	0	0	0	0	1

$$IM(5) = 0. \quad (5.2.6)$$

5.2.8 $\chi = (3, 1, 0, 0)$, $\mathcal{O} = B_2$

Dim.	s	G -sat.	Comp.
0	(0, 0, 0, 0)	0	1
1	(0, 1, 0, 0)	A_1	1
4	($\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}$)	A_1	1
5 _a	(1, 1, -1, -1)	A_2	1
5 _b	(1, -1, 0, 0)	\tilde{A}_1	$\mathbb{Z}/2\mathbb{Z}$
6	(3, 1, 0, 0)	B_2	$\mathbb{Z}/2\mathbb{Z}$

	0	1	4	5 _a	5 _{b,t}	5 _{b,s}	6 _t	6 _s
0	1	1	1	1	1	q ²	1	q ²
1	0	1	0	1	0	0	1	q ²
4	0	0	1	1	1	0	1	0
5 _a	0	0	0	1	0	0	1	0
5 _{b,t}	0	0	0	0	1	0	1	0
5 _{b,s}	0	0	0	0	0	1	0	1
6 _t	0	0	0	0	0	0	1	0
6 _s	0	0	0	0	0	0	0	1

$$IM(6_t) = 0, \quad IM(6_s) = 4. \quad (5.2.7)$$

5.2.9 $\chi = (2, 1, 1, 0)$, $\mathcal{O} = A_2 + \tilde{A}_1$

Dim.	s	G -sat.	Comp.
0	(0, 0, 0, 0)	0	1
4	(0, 0, 1, 0)	A_1	1
5	(0, 1, 1, 0)	\tilde{A}_1	$\mathbb{Z}/2\mathbb{Z}$
7	$(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, -\frac{1}{2})$	$A_1 + \tilde{A}_1$	1
8	$(1, -1, 1, 1)$	A_2	$\mathbb{Z}/2\mathbb{Z}$
9	(2, 1, 1, 0)	$A_2 + \tilde{A}_1$	1

	0	4	5_t	5_s	7	8_t	8_s	9
0	1	$1+q$	$1+q+q^2$	q	$1+q$	$1+q+q^2$	q	1
4	0	1	1	0	1	$1+q$	q	1
5_t	0	0	1	0	1	$1+q$	0	1
5_s	0	0	0	1	1	0	0	0
7	0	0	0	0	1	1	0	1
8_t	0	0	0	0	0	1	0	1
8_s	0	0	0	0	0	0	1	0
9	0	0	0	0	0	0	0	1

$$IM(9) = 0. \quad (5.2.8)$$

5.2.10 $\chi = (2, 0, 0, 0)$, $\mathcal{O} = A_2$

Dim.	s	G -sat.	Comp.
0	(0, 0, 0, 0)	0	1
7	$(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$	A_1	1
10	(1, -1, 0, 0)	\tilde{A}_1	$\mathbb{Z}/2\mathbb{Z}$
13	$(\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$	$A_1 + \tilde{A}_1$	1
14	(2, 0, 0, 0)	A_2	$\mathbb{Z}/2\mathbb{Z}$

	0	7	10_t	10_s	13	14_t	14_s
0	1	$1+q^3$	$1+q^3$	q^2	$1+q^3$	1	q^3
7	0	1	1	0	1	1	q^3
10_t	0	0	1	0	1	1	0
10_s	0	0	0	1	q	0	0
13	0	0	0	0	1	1	0
14_t	0	0	0	0	0	1	0
14_s	0	0	0	0	0	0	1

$$IM(14_t) = 0, \quad IM(14_s) = 10_s. \quad (5.2.9)$$

5.2.11 $\chi = (2, 2, 0, 0)$, $\mathcal{O} = \tilde{A}_2$

Dim.	s	G -sat.	Comp.
0	(0, 0, 0, 0)	0	1
7	(0, 1, -1, 0)	\tilde{A}_1	1
8	(2, 2, 0, 0)	\tilde{A}_2	1

	0	7	8
0	1	$1+q^3$	1
7	0	1	1
8	0	0	1

$$IM(8) = 0. \quad (5.2.10)$$

5.2.12 $\chi = (\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, $\mathcal{O} = A_1 + \tilde{A}_1$

Dim.	s	G -sat.	Comp.
0	(0, 0, 0, 0)	0	1
3	$(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	A_1	1
5	(1, 0, 0, 1)	\tilde{A}_1	$\mathbb{Z}/2\mathbb{Z}$
6	$(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	$A_1 + \tilde{A}_1$	1

	0	3	5_t	5_s	6
0	1	1	$1+q^2$	q	1
3	0	1	1	0	1
5_t	0	0	1	0	1
5_s	0	0	0	1	0
6	0	0	0	0	1

$$IM(6) = 0. \quad (5.2.11)$$

5.2.13 $\chi = (1, 1, 0, 0)$, $\mathcal{O} = \tilde{A}_1$

Dim.	s	G -sat.	Comp.
0	(0, 0, 0, 0)	0	1
6	(0, 1, 0, 0)	\tilde{A}_1	1
7	(1, 1, 0, 0)	\tilde{A}_1	$\mathbb{Z}/2\mathbb{Z}$

	0	6	7_t	7_s
0	1	1	1	q^3
6	0	1	1	0
7_t	0	0	1	0
7_s	0	0	0	1

$$IM(7_t) = 0, \quad IM(7_s) = 6. \quad (5.2.12)$$

5.2.14 $\chi = (1, 0, 0, 0)$, $\mathcal{O} = A_1$

Dim.	s	G -sat.	Comp.
0	(0, 0, 0, 0)	0	1
1	(1, 0, 0, 0)	A_1	1

	0	1
0	1	1
1	0	1

$$IM(1) = 0. \quad (5.2.13)$$

5.2.15 $\chi = (0, 0, 0, 0)$, $\mathcal{O} = 0$

There is only the trivial orbit, and only one polynomial equal to 1.

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