GENERIC UNIPOTENT STANDARD MODULES

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Abstract. Using the geometric classification, we find the reducibility points of a standard module for the affine Hecke algebra, in the case when the inducing data is generic. This recovers the known result of [MS] for representations of split p-adic groups with Iwahori-spherical Whittaker vectors. We also give a necessary (insufficient) condition for reducibility in the non-generic case.

By [L6], the unipotent representations of a split p-adic group $\mathcal{G}$ of adjoint type are classified in terms of geometric data for the dual complex group G. More precisely, they are indexed by certain triples $(\chi, \mathcal{O}, \mathcal{L})$, where χ is a Weyl orbit of semisimple elements in G, $\mathcal{O}$ is a "graded" orbit in the Lie algebra $\mathfrak{g}$, and $\mathcal{L}$ is a local system on $\mathcal{O}$. This is realized via equivalences with module categories for affine Hecke algebras of *geometric type* constructed from G ([L2, L3]). It is shown in [R], that in this correspondence, the unipotent representations of $\mathcal{G}$ admitting Whittaker vectors (*generic*) correspond to maximal orbits $\mathcal{O}$ and trivial $\mathcal{L}$. For Iwahori-spherical representations, the same result, with a different proof, follows from [BM1] (and [BM2]).

In this paper, we determine explicitly, as a consequence of the geometric classification, the reducibility points for the standard representations (in the sense of Langlands classification) when the inducing data is generic. This was known from [MS], as a consequence of the Langlands-Shahidi method. In particular, our main result, theorem 3.2 is essentially the same as proposition 3.3 in [MS] (our parameter ν corresponds to the parameter s in there). We also show that for non-generic inducing data, the reducibility points are necessarily a subset of those for the corresponding generic case.

For simplicity, we will work in the setting of the affine *graded Hecke algebra* of [L1], and real central character (section 1.2), from which one can recover the representation theory of the affine Hecke algebra (see section 4 in [L6] for example). Most of the paper is devoted to recording the relevant geometric results, particularly from [L7]. Once they are in place, the reducibility follows immediately by a simple comparison of dimensions of orbits.

The information about reducibility of standard modules played an important role in the determination of the generic Iwahori-spherical unitary dual (equivalently, spherical unitary dual) of split p-adic groups of exceptional types in [BC]. In fact, this paper is mainly motivated by that work.

1. Graded Hecke algebra

1.1. Let $\mathfrak{h}$ be a finite dimensional vector space, $R \subset \mathfrak{h}^*$ a root system, with $\Pi = \{\alpha_1, \dots, \alpha_n\}$ the set of simple roots, $\check{R} \subset \mathfrak{h}$ the set of coroots, and W the Weyl

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group. Let $c : R \to \mathbb{Z}_{>0}$ be a function such that $c_\alpha = c_\beta$, whenever α and β are W-conjugate. As a vector space,

$$\mathbb{H} = \mathbb{C}[W] \otimes \mathbb{A}, \tag{1.1.1}$$

where $\mathbb{A}$ is the symmetric algebra over $\mathfrak{h}^*$. The generators are $t_w \in \mathbb{C}[W]$, $w \in W$ and $\omega \in \mathfrak{h}^*$. The relations between the generators are:

$$\begin{aligned} &t_w t_{w'} = t_{ww'}, && \text{for all } w, w' \in W;\\ &t_s^2 = 1, && \text{for any simple reflection } s \in W;\\ &\omega t_s = t_s s(\omega) + c_\alpha \omega(\check\alpha), && \text{for simple reflections } s = s_\alpha. \end{aligned} \tag{1.1.2}$$

1.2. By [L1], the center of $\mathbb{H}$ is $\mathbb{A}^W$. On any simple (finite dimensional) $\mathbb{H}$-module, the center of $\mathbb{H}$ acts by a character, which we will call a *central character*. The central characters correspond to W-conjugacy classes of semisimple elements $\chi \in \mathfrak{h}$. We will assume throughout the paper that the characters are *real*, *i.e.*, hyperbolic.

1.3. We present the *Langlands classification* for $\mathbb{H}$ as in [E]. If V is a (finite dimensional) simple $\mathbb{H}$-module, $\mathbb{A}$ induces a generalized weight space decomposition $V = \bigoplus_{\lambda\in\mathfrak{h}} V_\lambda$. Call λ a *weight of* V if $V_\lambda \neq 0$.

Definition. *The irreducible module σ is called tempered if $\omega_i(\lambda) \le 0$, for every weight $\lambda \in \mathfrak{h}$ of σ and every fundamental weight $\omega_i \in \mathfrak{h}^*$, and in addition, λ is zero on the real span of the set $x \in \mathfrak{h}^*$ perpendicular on coroots. If σ is tempered, and $\omega_i(\lambda) < 0$, for all λ, ω_i as above, σ is called a discrete series.*

For every $\Pi_P \subset \Pi$, define $R_M \subset R$ to be the set of roots generated by Π_P, $\check R_M \subset \check R$ the corresponding set of coroots, and $W_P \subset W$ the corresponding Weyl subgroup. (The notation will make more sense in the sequel, when $P = MN$ will denote a parabolic subgroup of the complex reductive group G.)

Let $\mathbb{H}_M$ be the Hecke algebra attached to $(\mathfrak{h}, R_M)$. It can be regarded naturally as a subalgebra of $\mathbb{H}$.

Define $\mathfrak{t} = \{\nu \in \mathfrak{h} : \langle\alpha, \nu\rangle = 0, \text{ for all } \alpha \in \Pi_P\}$ and $\mathfrak{t}^* = \{\lambda \in \mathfrak{h}^* : \langle\lambda, \check\alpha\rangle = 0, \text{ for all } \alpha \in \Pi_P\}$. Then $\mathbb{H}_M$ decomposes as

$$\mathbb{H}_M = \mathbb{H}_{M_s} \otimes S(\mathfrak{t}^*),$$

where $\mathbb{H}_{M_s}$ is the Hecke algebra attached to $(\mathbb{C}\langle\Pi_P\rangle, R_M)$.

We will denote by $I(P, U)$ the induced module $I(P, U) = \mathbb{H} \otimes_{\mathbb{H}_M} U$.

Theorem ([E])**.**
(1) *Every irreducible $\mathbb{H}$-module is a quotient of a standard induced module $X(P, \sigma, \nu) = I(P, \sigma \otimes \mathbb{C}_\nu)$, where σ is a tempered module for $\mathbb{H}_{M_s}$, and $\nu \in \mathfrak{t}^+ = \{\nu \in \mathfrak{t} : \alpha(\nu) > 0, \text{ for all } \alpha \in \Pi \setminus \Pi_P\}$.*
(2) *Assume the notation from (1). Then $X(P, \sigma, \nu)$ has a unique irreducible quotient, denoted by $L(P, \sigma, \nu)$.*
(3) *If $L(P, \sigma, \nu) \cong L(P', \sigma', \nu')$, then $\Pi_P = \Pi_{P'}$, $\sigma \cong \sigma'$ as $\mathbb{H}_{M_s}$-modules, and $\nu = \nu'$.*

We will call a triple (P, σ, ν) as in theorem 1.3, a *Langlands parameter*.

2. Geometric parameterization

Notation. In the following, whenever Q denotes a complex Lie group, Q^0 will be the identity component, and $\mathfrak{q}$ will denote the Lie algebra. If s is an element of Q or $\mathfrak{q}$, we will denote by $Z_Q(s)$ the centralizer in Q of s.

2.1. Let G be a reductive connected complex algebraic group, with Lie algebra $\mathfrak{g}$. Let B be a Borel subgroup, and $A \subset B$ a maximal torus. Let $S = LU$ denote a parabolic subgroup, with $\mathfrak{s} = \mathfrak{l} + \mathfrak{u}$ the corresponding Lie algebras, such that $\mathfrak{l}$ admits an irreducible L-equivariant cuspidal local system (as in [L2],[L5]) Ξ on a nilpotent L-orbit $\mathcal{C} \subset \mathfrak{l}$. The classification of cuspidal local systems can be found in [L5]. In particular, $W = N(L)/L$ is a Coxeter group.

Let H be the center of L with Lie algebra $\mathfrak{h}$, and let R be the set of nonzero weights α for the ad-action of $\mathfrak{h}$ on $\mathfrak{g}$, and $R^+ \subset R$ the set of weights for which the corresponding weight space $\mathfrak{g}_\alpha \subset \mathfrak{u}$. For each parabolic $S_j = L_jU_j$, $j = 1, n$, such that $S \subset S_j$ maximally and $L \subset L_j$, let $R_j^+ = \{\alpha \in R^+ : \alpha(\mathfrak{z}(\mathfrak{l}_j)) = 0\}$, where $\mathfrak{z}(\mathfrak{l}_j)$ denotes the center of $\mathfrak{l}_j$. It is shown in [L2] that each R_j^+ contains a unique α_j such that $\alpha_j \notin 2R$.

Let $Z_G(\mathcal{C})$ denote the centralizer in G of a Lie triple for $\mathcal{C}$, and $\mathfrak{z}(\mathcal{C})$ its Lie algebra.

Proposition ([L2]). (1) *R is a (possibly non-reduced) root system in $\mathfrak{h}^*$, with simple roots $\Pi = \{\alpha_1, \dots, \alpha_n\}$, with Weyl group W.*
(2) *H is a maximal torus in $Z^0 = Z_G^0(\mathcal{C})$.*
(3) *W is isomorphic to $W(Z_G^0(\mathcal{C})) = N_{Z^0}(H)/H$.*
(4) *The set of roots in $\mathfrak{z}(\mathcal{C})$ with respect to $\mathfrak{h}$ is exactly the set of non-multipliable roots in R.*

For each $j = 1, \dots, n$, let $d_j \geq 2$ be such that

$$(ad(e)^{d_j-2} : \mathfrak{l}_j \cap \mathfrak{u} \to \mathfrak{l}_j \cap \mathfrak{u}) \neq 0, \text{ and } (ad(e)^{d_j-1} : \mathfrak{l}_j \cap \mathfrak{u} \to \mathfrak{l}_j \cap \mathfrak{u}) = 0. \tag{2.1.1}$$

By proposition 2.12 in [L2], $d_i = d_j$ whenever α_i and α_j are W-conjugate. Therefore, as in (1.1.1),(1.1.2), we can define a Hecke algebra $\mathbb{H}_S$ with parameters $c_j = d_j/2$. The explicit algebras which may appear are listed in 2.13 of [L2]. (The case of Hecke algebras with equal parameters arises when one takes $S = B$, and $\mathcal{C}$ and Ξ to be trivial.)

If $P \subset G$ is a parabolic subgroup, such that $S \subset P$, then denote

$$\Pi_{P/S} = \{\alpha_j \in \Pi : S_j \subset P\}. \tag{2.1.2}$$

When $S = B$, we write just Π_P.

Let us denote by $\Phi(G)$ the set of graded Hecke algebras $\mathbb{H}_S$ obtained by the above construction. The unique Hecke algebra with equal parameters in $\Phi(G)$ will be denoted by $\mathbb{H}_0$.

2.2. Let $\mathfrak{a}$ be the Lie algebra of the torus $A \subset G$. Fix a semisimple element, (*an infinitesimal character*) $\chi \in \mathfrak{a}$, and set

$$G_0 = \{g \in G : Ad(g)\chi = \chi\}, \quad \mathfrak{g}_n = \{y \in \mathfrak{g} : [\chi, y] = ny\}. \tag{2.2.1}$$

Whenever Q is a subgroup with Lie algebra $\mathfrak{q}$, we will write $Q_0 = Q \cap G_0$ and $\mathfrak{q}_n = \mathfrak{q} \cap \mathfrak{g}_n$.

For $\mathbb{H} \in \Phi(G)$, corresponding to $S = LU$ in the notation of section 2.1, denote by $\mathrm{mod}_\chi \mathbb{H}$ the category of finite dimensional $\mathbb{H}$-modules of central character congruent to χ modulo $\mathfrak{z}(\mathfrak{l})$.

Theorem ([L3])**.** *There exists a one-to-one correspondence between the standard (or irreducible) objects in $\sqcup_{\mathbb{H}\in\Phi(G)} mod_\chi(\mathbb{H})$ and the set of pairs $\xi = (\mathcal{O}, \mathcal{L})$, where*

(1) *$\mathcal{O}$ is a G_0-orbit on $\mathfrak{g}_1$.*
(2) *$\mathcal{L}$ is an irreducible G_0-equivariant local system on $\mathcal{O}$.*

We say that two modules in $\sqcup_{\mathbb{H}\in\Phi(G)} \mathrm{mod}_\chi(\mathbb{H})$ are in the same *L-packet* if they correspond to the same orbit $\mathcal{O}$.

For the $\mathbb{H}_0$-modules, the local systems which appear are of *Springer type* ([L7]). More precisely, if $e \in \mathcal{O}$, then $\mathcal{L}$ corresponds to a representation ϕ of the component group $Z_{G_0}(e)/Z_{G_0}(e)^0$. The representations ϕ which are allowed must be in the restriction $Z_{G_0}(e)/Z_{G_0}(e)^0 \subset Z_G(e)/Z_G(e)^0$ of a representation which appears in Springer's correspondence.

2.3. Let $Orb_n(\chi)$ denote the set of G_0 orbits on $\mathfrak{g}_n$, $n \in \mathbb{Z} \setminus \{0\}$.

(1) $Orb_n(\chi)$ is finite.
(2) For every $\mathcal{O} \in Orb_n(\chi)$, $\overline{\mathcal{O}} \setminus \mathcal{O}$ is the union of some orbits $\mathcal{O}'$ with $\dim \mathcal{O}' < \dim \mathcal{O}$.
(3) There is a unique open (dense) orbit $\mathcal{O}_{open}$ in $Orb_n(\chi)$.

A parameterization for $Orb_n(\chi)$ appeared in [K]. We will instead use, in sections 2.6 and 2.7, the formulation of [L7].

2.4. By [L4], the categories $\mathrm{mod}_\chi \mathbb{H}$, $\mathbb{H} \in \Phi(G)$, have tempered modules if and only if χ is the middle element of a nilpotent orbit in $\mathfrak{g}$. In this case the standard modules parameterized by $(\mathcal{O}_{open}, \mathcal{L})$ are irreducible and they exhaust the tempered modules. If in addition, χ is the middle element of a distinguished nilpotent orbit, then the tempered modules are discrete series.

2.5. By [R], there is a unique *generic* module in $\sqcup_{\mathbb{H}\in\Phi(G)} \mathrm{mod}_\chi(\mathbb{H})$, which is parametrized by $(\mathcal{O}_{open}, triv)$, where *triv* denotes the trivial local system. Note that this is always a module of $\mathbb{H}_0$. The fact that the generic module in $\mathrm{mod}_\chi(\mathbb{H}_0)$ is parameterized by $(\mathcal{O}_{open}, triv)$ is also an immediate consequence of the results in [BM1] and [BM2]. In [BM1], it is proven that the generic $\mathbb{H}_0$-module is characterized by the property that it contains the *sign* representation of W.

2.6. Let e be a representative of an orbit $\mathcal{O} = \mathcal{O}_e$ in $\mathfrak{g}_1$. To e, one associates, conform [L7], a parabolic subalgebras of $\mathfrak{g}$, which we'll denote $\mathfrak{p}^e$. This is used to give a parameterization of $Orb_1(\chi)$.

By the graded version of the Jacobson-Morozov triple ([L7]), $e \in \mathfrak{g}_1$ can be embedded into a Lie triple $\{e, h, f\}$, such that $h \in \mathfrak{g}_0$, and $f \in \mathfrak{g}_{-1}$. Define a gradation of $\mathfrak{g}$ with respect to h as well,

$$\mathfrak{g}^r = \{y \in \mathfrak{g} : [h, y] = 2ry\}, \tag{2.6.1}$$

and set

$$\mathfrak{g}_t^r = \mathfrak{g}_t \cap \mathfrak{g}^r. \tag{2.6.2}$$

Then

$$\mathfrak{g} = \bigoplus_{t,r} \mathfrak{g}_t^r. \tag{2.6.3}$$

Set

$$\mathfrak{m}^e = \bigoplus_{t=r} \mathfrak{g}_t^r, \quad \mathfrak{n}^e = \bigoplus_{t<r} \mathfrak{g}_t^r, \quad \mathfrak{p}^e = \mathfrak{m}^e \oplus \mathfrak{n}^e. \tag{2.6.4}$$

Definition. *One says that χ is rigid for a Levi subalgebra $\mathfrak{m}$, if χ is congruent modulo $\mathfrak{z}(\mathfrak{m})$ to a middle element of a nilpotent orbit in $\mathfrak{m}$.*

Proposition ([L7])**.** *Consider the subalgebra $\mathfrak{p}^e$ defined by (2.6.4), and let P^e be the corresponding parabolic subgroup.*

1. *$\mathfrak{p}^e$ depends only on e and not on the entire Lie triple $\{e, h, f\}$.*
2. *χ is rigid for $\mathfrak{m}^e$. In particular, e is in the open M_0-orbit in $\mathfrak{m}_1^e$.*
3. *The P_0^e-orbit of e in $\mathfrak{p}_1^e$ is open, dense in $\mathfrak{p}^e$.*
4. *$Z_{G_0}(e) \subset P^e$, and the inclusion $Z_{M_0}(e) \subset Z_{G_0}(e)$ induces an isomorphism of the component groups.*

An immediate corollary of (3) and (4) in the proposition is a dimension formula for the orbits in $Orb_1(\chi)$.

Corollary ([L7])**.** *For an orbit $\mathcal{O}_e \in Orb_1(\chi)$,*

$$\dim \mathcal{O}_e = \dim \mathfrak{p}_1^e - \dim \mathfrak{p}_0^e + \dim \mathfrak{g}_0, \tag{2.6.5}$$

where $\mathfrak{p}_i^e = \mathfrak{p}^e \cap \mathfrak{g}_i$.

2.7. **Definition.** *A parabolic subgroup P with Lie algebra $\mathfrak{p}$ is called good for χ if $\mathfrak{p} = \mathfrak{p}^e$ for some nilpotent $e \in \mathfrak{g}_1$ (notation as in (2.6.4)), and such that it satisfies (2) in proposition 2.6.*

Let $\mathcal{P}(\chi)$ denote the set of good parabolic subgroups for χ.

Theorem ([L7])**.** *The map $\mathcal{O}_e \mapsto P^e$ defined in section 2.6 induces a bijection between $Orb_1(\chi)$ and G_0-conjugacy classes in $\mathcal{P}(\chi)$.*

Proof. We record the definition of the inverse map. Let $P = MN$ be a good parabolic for χ. Then there exists s a middle element of a Lie triple in $\mathfrak{m}$, such that $\chi \equiv s \pmod{\mathfrak{z}(\mathfrak{m})}$. Moreover, the decomposition (2.6.4) must hold with respect to χ and s. Let $G_0^0 \subset G_0$ be the reductive subgroup whose Lie algebra is $\mathfrak{g}_0^0$ (notation as in section 2.6). Then G_0^0 acts on $\mathfrak{g}_1^1$, and there is a unique open orbit of this action. Let $\mathcal{O}$ be the unique G_0-orbit on $\mathfrak{g}_1$ containing it. The inverse map associates $\mathcal{O}$ to P. □

3. Reducibility points

3.1. Let $\{e, h, f\}$ be a graded Lie triple for the orbit $\mathcal{O}_e \in Orb_1(\chi)$. Assume that $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ is a standard parabolic subalgebra, $\mathfrak{b} \subset \mathfrak{p}$, such that $\{e, h, f\} \subset \mathfrak{m}$. Let $\bar{\mathfrak{p}} = \mathfrak{m} + \bar{\mathfrak{n}}$ be the opposite parabolics subalgebra. Let $\Pi_P \subset \Pi$ denote the simple roots defining P, and denote by R_M and R_N the roots in $\mathfrak{m}$, respectively $\mathfrak{n}$. We can write $\chi = \frac{1}{2}h + \underline{\nu}$, with $\underline{\nu} \in \mathfrak{z}_G(e, h, f)$.

Lemma. *If $\chi = \frac{1}{2}h + \nu$ has ν dominant with respect to R_N, then $\bar{\mathfrak{p}}$ is a good parabolic for χ. In fact, $\bar{\mathfrak{p}} = \mathfrak{p}^e$.*

Proof. From (2.6.4) and the dominance conditions, we see immediately that $\mathfrak{m} = \mathfrak{m}^e$, and $\bar{\mathfrak{n}} = \mathfrak{n}^e$. □

Let σ be the tempered module of $\mathbb{H}_{M_s}$ (notation as in 1.3) parameterized by $\{e,h,f\}$. By the induction theorem from [L4], we known then that, in the correspondences of theorems 2.2 and 2.7, the standard module $X(P,\sigma,\nu)$, and the Langlands quotient $L(P,\sigma,\nu)$ are parameterized in $Orb_1(\chi)$ by the parabolic subalgebra $\bar{\mathfrak{p}}$.

3.2. Now assume that $\mathfrak{p}=\mathfrak{m}+\mathfrak{n}$ is a *maximal* parabolic of $\mathfrak{g}$. Then $\Pi\setminus\Pi_P=\{\alpha\}$. Let $\check\omega$ denote the fundamental coweight for α.

As before, let σ be the tempered module attached to the map

$$\mathfrak{s}l(2)=\mathbb{C}\langle e,h,f\rangle\hookrightarrow\mathfrak{m}. \tag{3.2.1}$$

Then $\mathfrak{n}$ is an $\mathfrak{s}l(2)$-module, via the adjoint action of $\mathfrak{m}$. Let $k(\alpha)$ denote the multiplicit with which α appears in the highest root for R. [1]

The coweight $\check\omega$ commutes with the $\mathfrak{s}l(2)$. Decompose $\mathfrak{n}$ as $\mathfrak{n}=\oplus_{i=1}^{k(\alpha)}\mathfrak{n}_i$, where $\mathfrak{n}_i$ is the i-eigenspace of $\check\omega$. Then decompose each $\mathfrak{n}_i$ into simple $\mathfrak{s}l(2)$-modules

$$\mathfrak{n}_i=\oplus_j(d_{ij}),\ i=1,\dots,k(\alpha), \tag{3.2.2}$$

where (d) is the simple $\mathfrak{s}l(2)$-module of dimension d.

Theorem. *Let $\mathfrak{p}=\mathfrak{m}+\mathfrak{n}$ be a maximal parabolic, and σ be a generic tempered module parameterized by (3.2.1). Then the reducibility points $\nu>0$ of the standard $\mathbb{H}_0$-module $X(P,\sigma,\nu)$ are*

$$\nu\in\left\{\frac{d_{ij}+1}{2i}\right\}_{i,j}, \tag{3.2.3}$$

where the integers d_{ij} are defined in (3.2.2). Equivalently, these are the zeros of the rational function in ν,

$$\prod_{\beta\in R_N}\frac{1-\langle\beta,\chi\rangle}{\langle\beta,\chi\rangle}, \tag{3.2.4}$$

where $\chi=\frac{1}{2}h+\nu\check\omega$ is the infinitesimal character of $X(P,\sigma,\nu)$.

Proof. Let $\mathcal{O}(\bar{\mathfrak{p}})$ be the orbit parameterizing $X(P,\sigma,\nu)$ by section 3.1. By section 2.5, $X(P,\sigma,\nu)$ is irreducible if and only if $\mathcal{O}(\bar{\mathfrak{p}})=\mathcal{O}_{open}$.

Corollary 2.6 implies that $\dim\mathcal{O}(\bar{\mathfrak{p}})=\dim\mathfrak{g}_0-\dim(\mathfrak{g}_0\cap\bar{\mathfrak{p}})+\dim(\mathfrak{g}_1\cap\bar{\mathfrak{p}})$. From this and the fact that $\dim\mathcal{O}_{open}=\dim\mathfrak{g}_1$, it follows that $\mathcal{O}(\bar{\mathfrak{p}})=\mathcal{O}_{open}$ if and only if

$$\#\{\beta\in R_N:\langle\beta,\chi\rangle=1\}=\#\{\beta\in R_N:\langle\beta,\chi\rangle=0\}. \tag{3.2.5}$$

Consider the rational function of ν, $\prod_{\beta\in R_N}\frac{1-\langle\beta,\chi\rangle}{\langle\beta,\chi\rangle}$. Therefore, the reducibility points are given by the zeros of this function.

The explicit list of reducibility points follows from the fact that $\{\langle\beta,h\rangle:\beta\in R_N\}=\sqcup_{i,j}\{d_{ij}-1,d_{ij}-3,\dots,-d_{ij}+1\}$, and so

$$\prod_{\beta\in R_N}\frac{1-\langle\chi,\beta\rangle}{\langle\chi,\beta\rangle}=\prod_{i,j}\frac{\frac{d_{ij}+1}{2i}-\nu}{\frac{d_{ij}-1}{2i}+\nu}. \tag{3.2.6}$$

□

[1] If $\mathfrak{g}$ is a classical simple algebra, this multiplicity is always 1 or 2.

We remark that in the proof of formula (3.2.4), one does not use the assumption that $\mathfrak{p}$ be maximal parabolic. This formula holds as is for any parabolic $\mathfrak{p}$.

Example. The most interesting example of reducibility points for maximal parabolic induction is the case $\Pi_P = A_4 + A_2 + A_1$ in $\Pi = E_8$, with e the principal nilpotent in $A_4 + A_2 + A_1$ (which means that σ is the Steinberg representation). Then $k(\alpha) = 6$, $\dim \mathfrak{n} = 106$, and the $sl(2)$ decompositions (3.2.2) are

$$\begin{aligned} &\mathfrak{n}_1 = (8) + 2\cdot(6) + 2\cdot(4) + (2) && \mathfrak{n}_2 = (9) + (7) + 2\cdot(5) + (3) + (1) \\ &\mathfrak{n}_3 = (8) + (6) + (4) + (2) && \mathfrak{n}_4 = (7) + (5) + (3) \\ &\mathfrak{n}_5 = (4) + (2) && \mathfrak{n}_6 = (5). \end{aligned} \tag{3.2.7}$$

There are 11 reducibility points:

$$\left\{\frac{3}{10}, \frac{1}{2}, \frac{3}{4}, \frac{5}{6}, 1, \frac{7}{6}, \frac{3}{2}, 2, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}\right\}. \tag{3.2.8}$$

3.3. One also immediately obtains a partial result for non-generic data. Recall the notation and construction of section 2.1. In particular, if σ' is parameterized by (3.2.1), there exists a unique triple $(S, \mathcal{C}, \Xi)$ such that σ' is a discrete series for the subalgebra $\mathbb{H}_{S,\Pi_{P/S}}$ in $\mathbb{H}_S$.

Proposition. *Let σ and σ' be tempered modules in the L-packet parameterized by (3.2.1), and assume that σ is generic. The standard $\mathbb{H}_S$-module $X(P/S, \sigma', \nu)$ is reducible for $\nu > 0$ only if the standard $\mathbb{H}_0$-module $X(P, \sigma, \nu)$ is reducible.*

Proof. If $X(P/S, \sigma', \nu)$ is reducible, then the corresponding orbit is not the open orbit. But this means $X(P, \sigma, \nu)$ is reducible as well. □

Remark. This result gives necessary conditions for reducibility, but not sufficient. In fact, they are far from being sharp for non-generic inducing data as seen in the following example.

Example. Consider $\mathbb{H}_0$ of type C_{n+1}, and $\mathfrak{p}$ of type C_n, and assume that n is a triangular number. Let the nilpotent element e correspond to the distinguished orbit $(2, 4, \dots, 2k)$ in $\mathfrak{sp}(2n)$, and χ be half the middle element of a Lie triple for e.

There are $\binom{k}{\lfloor \frac{k}{2} \rfloor}$ discrete series in $\mathrm{mod}_\chi \mathbb{H}_0(C_n)$. Let σ be the generic one. There exists a discrete series, call it σ', characterized by the fact that $\sigma'|_{W(C_n)} = \mu_k$, where

$$\mu_k = \begin{cases} m^{2m+1} \times 0, & \text{if } k = 2m \\ 0 \times (m+1)^{2m+1}, & \text{if } k = 2m+1. \end{cases} \tag{3.3.1}$$

(The notation for $W(C_n)$-representations, and the algorithms necessary for the computation are as in [L5].)

Theorem 3.2 implies that the reducibility points, $\nu > 0$, for $X(C_n, \sigma, \nu)$ are

$$\nu \in \left\{\frac{1}{2},\ \frac{3}{2},\ \frac{5}{2}, \dots,\ k + \frac{1}{2}\right\},$$

but one can show that the reducibility points of $X(C_n, \sigma', \nu)$ are just

$$\nu \in \left\{\lfloor \frac{k}{2} \rfloor + \frac{1}{2},\ k + \frac{1}{2}\right\}.$$

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