SPHERICAL UNITARY REPRESENTATIONS FOR REDUCTIVE GROUPS

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1. Classical motivation: spherical functions

1.1. **Spherical harmonics.** Let $S^{n-1} \subset \mathbb{R}^n$ be the $(n-1)$-dimensional sphere, $C^\infty(S^{n-1})$ the set of smooth complex functions, and L the Laplacian operator (e.g., in S^2, $L = \frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\psi^2}$).

Then L acts on $C^\infty(S^{n-1})$ and the action is diagonalizable.

Problem: Describe the eigenspaces E_λ.

Solution: If $O(n) = \{A \in GL(n,\mathbb{R}) : A \cdot A^t = I_n\}$ is the orthogonal group, then

$$S^{n-1} = O(n)/O(n-1),$$

($O(n-1)$ is the isotropy group of the north pole o) and the action of $O(n)$ commutes with that of L, so $O(n)$ acts on each E_λ, so one can use representation theoretic tools to describe E_λ. (In fact, L generates the algebra of $O(n)$-invariant differential operators on S^{n-1}.)

Definition 1.1. *A representation (π, V) of G on a locally convex topological space V is a continuous homomorphism $\pi : G \times V \to V$.*

Theorem 1.2.

(1) *The eigenvalues are $\{\lambda_k = -k(k+n-2) : k \in \mathbb{N}\}$. The eigenfunctions are the spherical harmonics, i.e., $p\,|_{S^{n-1}}$, where p is a harmonic polynomial of degree k.*
(2) *Each E_k is irreducible as an $O(n)$-representation.*
(3) $L^2(S^{n-1}) = \oplus_{k=0}^\infty E_k$.
(4) *Each E_k contains a unique element ϕ_k invariant under $O(n-1)$, $\phi_k(o) = 1$.*

Note that for $n = 2$, this is just usual Fourier analysis.

1.2. **Spherical functions.** Let G be a connected real Lie group, K a compact subgroup. Let $D(G/K)$ be the algebra of left G-invariant differential operators on $X = G/K$.

Definition 1.3. *A function $\phi \in C^\infty(G/K)$ is called spherical if:*

(1) *ϕ is an eigenfunction for all operators in $D(G/K)$.*
(2) *ϕ is left K-invariant.*

Date: January 23, 2007.

In particular, one can think of ϕ as a function in $C^\infty(G)$ biinvariant under K. Therefore, define $\mathcal{H}(G,K)$ to be the algebra (under convolution) of compactly supported K-biinvariant functions on G.

The case we are interested in is when $\mathcal{H}(G,K)$ is commutative, in particular when G is a *semisimple* connected Lie group (locally isomorphic with a product of simple Lie groups), and K is a maximal compact subgroup. For example:

$$\begin{aligned} G &= SL(2,\mathbb{R}) \\ K &= SO(2) \\ G/K &= \{z\in\mathbb{C} : Im\ z>0\}. \end{aligned}$$

In this case, $D(G/K)$ is a commutative polynomial ring in $rank(G)$ generators.The problem is to determine the joint eigenspaces $C^\infty(G/K)_\lambda$ of $D(G/K)$. The eigenvalues are indexed by $\lambda\in\mathfrak{a}_{\mathbb{C}}^*$ (see section 2.1), and every eigenspace contains a unique spherical function.

Definition 1.4. *If the space V is a Hilbert space, and G acts by unitary operators, then (π,V) is called a unitary representation.*

Theorem 1.5. *For every irreducible representation (π,V) of G,* $\dim\pi^K\le 1$.

A representation (π,V) is called *spherical* if there exists $e\in V$, such that $\pi(k)e=e$, for all $k\in K$.

A function $\phi\neq 0$ is called *positive definite* if

$$\sum_{i,j=1}^{n}\phi(x_i^{-1}x_j)\alpha_i\bar{\alpha}_j\ge 0,$$

for all $x_1,\dots,x_n\in G$, $\alpha_1,\dots,\alpha_n\in\mathbb{C}$. A positive definite function is necessarily *bounded* because $|\phi(x)|\le\phi(1)$ for all $x\in G$. Clearly, the function $\phi\equiv 1$ is positive definite.

Theorem 1.6 (Gelfand-Naimark, Godement)**.** *There is a one-to-one correspondence:*

$$\left\{\begin{matrix}\textit{nonzero positive definite}\\ \textit{spherical functions on } G\end{matrix}\right\}\leftrightarrow\left\{\begin{matrix}\textit{equivalence classes of irreducible}\\ \textit{unitary spherical } G\textit{-reps } (\pi,V)\end{matrix}\right\}.$$

In one direction, if (π,V) is a unitary representation, with e a unit spherical vector, the *matrix coefficient*

$$x\mapsto\langle e,\pi(x)e\rangle,\ x\in G$$

is a positive definite spherical function.

Denote

$$\begin{aligned}\Pi^s(G) &= \text{ equiv. classes of spherical irreducible } G\text{-representations},\\ \Pi^s_u(G) &= \text{ unitary representations in } \Pi^s(G).\end{aligned}$$

Problem. Determine $\Pi^s_u(G)$.

Always, one element in $\Pi^s_u(G)$ is the *trivial representation*. It corresponds to $\phi \equiv 1$.

2. Spherical unitary dual

Let $\mathbb{F}$ be a local field of characteristic 0: $\mathbb{F} = \mathbb{R}, \mathbb{Q}_p$. The field $\mathbb{Q}_p$ of *p-adic numbers* is the completion of $\mathbb{Q}$ with respect to the p-adic norm. It contains the ring of *p-adic integers* $\mathbb{Z}_p$.

Let G be an algebraic subgroup of $GL_n(\mathbb{F})$, which is *simple* and *split*. The second condition means that G contains a maximal torus A (connected, commutative, diagonalizable), which is split, i.e. isomorphic to $(\mathbb{F}^*)^n$. For example, if $G = SL(n, \mathbb{F})$, A is the subgroup of diagonal matrices. More explicitly, the groups one is interested in are the *classical groups*: $SL(n, \mathbb{F})$, $Sp(2n, \mathbb{F})$, $SO(n, \mathbb{F})$ and the *exceptional groups*: G_2, F_4, $E_{6,7,8}$ (the index describes the rank of the maximal torus).

We also fix a special maximal compact subgroup. This depends on the field $\mathbb{F}$. In the $SL(n, \mathbb{F})$ case:

$$
\begin{aligned}
&\mathbb{F} = \mathbb{R} &\qquad K = SO(n)\\
&\mathbb{F} = \mathbb{Q}_p &\qquad K = SL(n, \mathbb{Z}_p).
\end{aligned}
$$

2.1. **Langlands classification.** The classification of $\Pi^s(G)$ is a particular case of the *Langlands classification*. We recall it now.

Let $B = AN$ be a Borel subgroup, A is the maximal split torus, N is a *unipotent* group. For $SL(n, \mathbb{F})$, B is the subgroup of upper triangular matrices, and N is the subgroup of upper triangular matrices with 1 on the diagonal.

The group A is abelian, so its representations are 1-dimensional, *quasicharacters* $\mathbb{F}^* \to \mathbb{C}^*$. For our problem, it is sufficient to consider only quasicharacters

$$| \, |^s : \mathbb{F}^* \to \mathbb{C}^*, \quad s \in \mathbb{C}.$$

($| \, |$ denotes the usual absolute value in $\mathbb{R}$ or $\mathbb{Q}_p$.) Such a quasicharacter is *unitary* only for $s \in i\mathbb{R}$.

Using $A \cong (F^*)^n$, we can define for each $\nu \in \mathbb{C}^n$, $\nu = (\nu_1, \dots, \nu_n)$ a character

$$\chi_\nu = | \, |^\nu : \mathbb{A} \to \mathbb{C}^*, \quad \chi_\nu(a_1, \dots, a_n) = \prod_{j=1}^n |a_j|^{\nu_j}.$$

A better way of thinking about ν, is that $\nu \in \mathfrak{a}^*_{\mathbb{C}}$, the Cartan subalgebra of the Lie algebra of the dual complex group $\check{G}$. (For example, if $G = Sp(2n, \mathbb{R})$, then $\check{G} = SO(2n+1, \mathbb{C})$. Let $\Delta \subset \mathfrak{a}^*_{\mathbb{R}}$ be the set of roots of A in G, and Δ^+ the positive roots given by B. Set

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha.$$

The character χ_ν can be thought of as a character of B, on which N acts trivially. We can form the induced module, the *spherical principal series*:

$$I(\nu) = Ind_B^G(\chi_\nu) = \{f : G \to \mathbb{C}^* : f(ang) = \chi_{\nu+\rho}(a)f(g)\},$$

on which G acts by right translations. The requirement is that the functions are smooth in the real case, and locally constant in the p-adic case. The role of the shift by ρ is so that the induction preserves unitarity. In this case this means $I(0)$ is unitary.

Note that at $\nu = -\rho$, the constant functions form a one-dimensional subspace, so the trivial representation appears as a submodule. But also at $\nu = \rho$, the trivial representation appears, but as a quotient.

Properties.

(1) $I(\nu)$ has a finite composition series
$$0 = V_0 \subset V_1 \subset \cdots \subset V_{r-1} \subset V_r = I(\nu),$$
V_j are submodules and V_j/V_{j-1} are the irreducible *subquotients*.
(2) There is a unique index j, such that $V_j/V_{j-1} = J(\nu)$ is *spherical*. This is the *Langlands subquotient*.
(3) Let $W = N_G(A)/A$ be the Weyl group. For every $w \in W$, $I(\nu)$ and $I(w\nu)$ have isomorphic irreducible composition factors with the same multiplicity. In particular, $J(\nu) \cong J(w\nu)$.
(4) Every element of $\Pi^s(G)$ appears in this way.

There is a reduction of unitarity to the case when ν is *real*. By (3), we can assume that ν is in the *dominant Weyl chamber* $\mathfrak{a}^*_{\mathbb{R},+}$, that is $\langle \check{\alpha}, \nu\rangle \geq 0$, for all $\alpha \in \Delta^+$. (Note that $\check{\alpha} \in \mathfrak{a}$ is the coroot associated to α.)
Remark. When λ is dominant and $\mathbb{F} = \mathbb{R}$, a conjecture of Helgason, proved by Kashiwara-K-M-O-O-T (1978) says that

$$I(\nu) \cong C^\infty(G/K)_\nu,$$

and the isomorphism is given by the *Poisson transform* $(P_\nu f)(gK) = \int_K f(gk)\,dk$.

A well-known result (Harish-Chandra) is that $I(\nu)$ is irreducible unless $\langle \check{\alpha}, \nu\rangle = m$, for $\alpha \in \Delta^+$, and certain positive integers m (for p-adic $m = 1$, for real $m = 1, 3, 5, ...$).

Define
$$SU(G(\mathbb{F})) = \{\nu \in \mathfrak{a}^*_{\mathbb{R},+} : J(\nu) \text{ is (pre)unitary}\}.$$

Equivalent problem. Determine $SU(G(\mathbb{F}))$.

[Insert picture of B_2]

One result from spherical functions: Helgason-Johnson (1969) showed that $SU(G(\mathbb{F}))$ is contained in the convex hull of ρ using the boundedness condition on positive definite spherical functions.

Obviously, $\rho \in SU(G(\mathbb{F}))$. This is the trivial representation. Moreover, if the rank of G is at least 2, it is *isolated*. (This is *Kazhdan's property T*.)

There are many partial results on the determination of $SU(G(\mathbb{F}))$. Complete results were obtained by Vogan ('86), Tadic ('86), Barbasch, Moy ('89, '95) for the classical groups.

2.2. **Motivation: automorphic forms.** Let G be defined over $\mathbb{Q}$. Let $\mathbb{A}$ be the locally compact ring of *adeles*, a restricted product of all completions of $\mathbb{Q}$, in which $\mathbb{Q}$ is embedded diagonally. The group $G(\mathbb{A})$ has $G(\mathbb{Q})$ as a discrete subgroup. One considers the space of square integrable functions (right-invariant measure)

$$L^2(G(\mathbb{Q})\backslash G(\mathbb{A})),$$

as a representation of $G(\mathbb{A})$ (by right translations). This is a unitary representation of $G(\mathbb{A})$. The main question is to decompose it into irreducible representations.

It is well-known that any such representation π has a unique decomposition

$$\pi = \pi_\infty \otimes \bigotimes_p \pi_p,$$

where π_∞ is a $G(\mathbb{R})$-representation and π_p is a $G(\mathbb{Q}_p)$-representation. All such *local factors* are unitary. Moreover, except for finitely many primes p, the representation π_p is *spherical*.

From a local perspective, the question is which representations of $G(\mathbb{F})$ appear in these decompositions. Arthur's conjectures refer to the *residual* part of the *discrete spectrum* of $G(\mathbb{A})$.

We recall one particular instance next. It will serve both as a motivation and as a indication of how the spherical unitary dual problem will be solved.

2.3. **Special unipotent representations.** Let $\check{\mathfrak{g}}$ be the dual Lie algebra. The $\check{G}$-conjugacy classes of homomorphisms

$$\phi : sl(2,\mathbb{C}) \to \check{\mathfrak{g}}$$

were classified by Dynkin in 1950's. Set the following notation: $\phi\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right) = \check{h}$, $\phi\left(\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right) = \check{e}$, $\phi\left(\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right) = \check{f}$. The set $\{\check{e},\check{h},\check{f}\}$ is called a *Lie triple* or a *Jacobson-Morozov triple*.

Denote

$$\mathcal{D}(\check{\mathfrak{g}}) = \text{ set of all } \check{h} \text{ appearing in this way.}$$

This is a finite subset of $\mathfrak{a}^*_{\mathbb{R},+}$. It contains in particular 0 and 2ρ.

Conjecture 2.1 (Arthur)**.** $\frac{1}{2}\mathcal{D}(\check{\mathfrak{g}}) \subset SU(G(\mathbb{F}))$.

This is a theorem for $\mathbb{F}$ p-adic (Barbasch-Moy) and for real classical groups (Barbasch).

2.4. **Results.**

Theorem 2.2 (Barbasch, Ciubotaru).
1. Assume $\mathbb{F}$ is p-adic. The set $SU(G(\mathbb{F}))$ can be described as follows: for every $\check{h} \in \mathcal{D}(\check{\mathfrak{g}})$, one attaches a complementary series $CS(G(\mathbb{F}), h)$ such that

$$SU(G(\mathbb{F})) = \sqcup_{\check{h}\in\mathcal{D}(\check{\mathfrak{g}})} CS(G(\mathbb{F}), \check{h}),$$

where

a) $CS(G(\mathbb{F}), 0)$ is a union of 2^k simplices (described explicitly) (e.g.: 16 in E_8).

b) If $\check{h} \in \mathcal{D}(\check{\mathfrak{g}})$, $CS(G(\mathbb{F}), \check{h}) = CS(H_{\check{h}}(\mathbb{F}), 0)$, where $H_{\check{h}}(\mathbb{F})$ is the split group whose dual complex group is the identity component of $Cent_{\check{G}}\{e, h, f\}$, with a few exceptions when $\Delta \in \{F_4, E_7, E_8\}$:

Δ	F_4	E_7	E_8
$\lvert\mathcal{D}(\check{\mathfrak{g}})\rvert$	16	46	70
Exceptions	1	1	6

2. If $\mathbb{F} = \mathbb{R}$, then $SU(G(\mathbb{R})) \subseteq SU(G(\mathbb{Q}_p))$.

We say a few words about the elements involved in this proof. Assume $\mathbb{F}$ is p-adic. Then by the Borel-Casselman correspondence (1976), there is an equivalence of categories between the category of subquotients of $I(\nu)$ on one hand, and finite dimensional modules of the Iwahori-Hecke algebra $\mathcal{H}(G, \mathcal{I})$ on the other. Barbasch and Moy (1989) showed the *preservation of unitarity*, i.e., the unitarity can be detected on the $\mathcal{H}(G, \mathcal{I})$-modules.

The dual of $\mathcal{H}(G, \mathcal{I})$ was classified geometrically by Kazhdan and Lusztig (1987): the triples $\{\check{e}, \check{h}, \check{f}\}$ appear naturally in this classification.

The general method to show that a module is non-unitary is by a computation of signatures of hermitian forms and intertwining operators on K-representations (Vogan).

Now we have a "machinery" for matching directly signatures for hermitian forms on $G(\mathbb{F})$-representations (or rather Hecke algebra modules) with signatures on $H_{\check{h}}(\mathbb{F})$. To finish the proof, we need to show that the remaining modules (not ruled out by this method) are indeed unitary. Here we use a combination of methods: unitary induction, deformation (complementary series), irreducibility, composition series.

Part 2 is based on D. Vogan's idea of *petite K-types*. Conjecturally, the two sets should be equal. For classical groups this is a theorem of D. Barbasch.

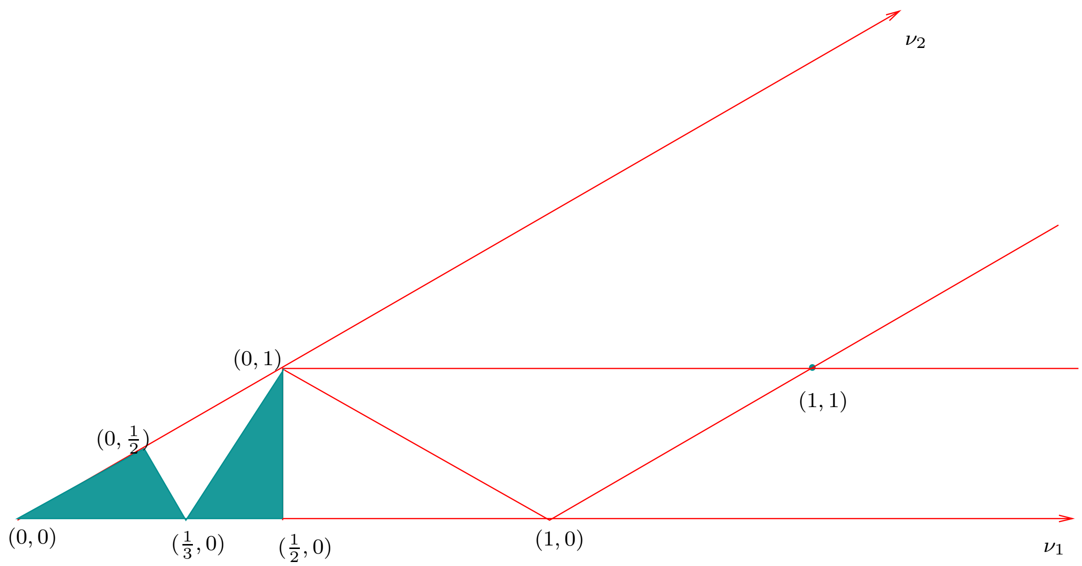

FIGURE 1. $SU(G_2)$ in the dominant Weyl chamber

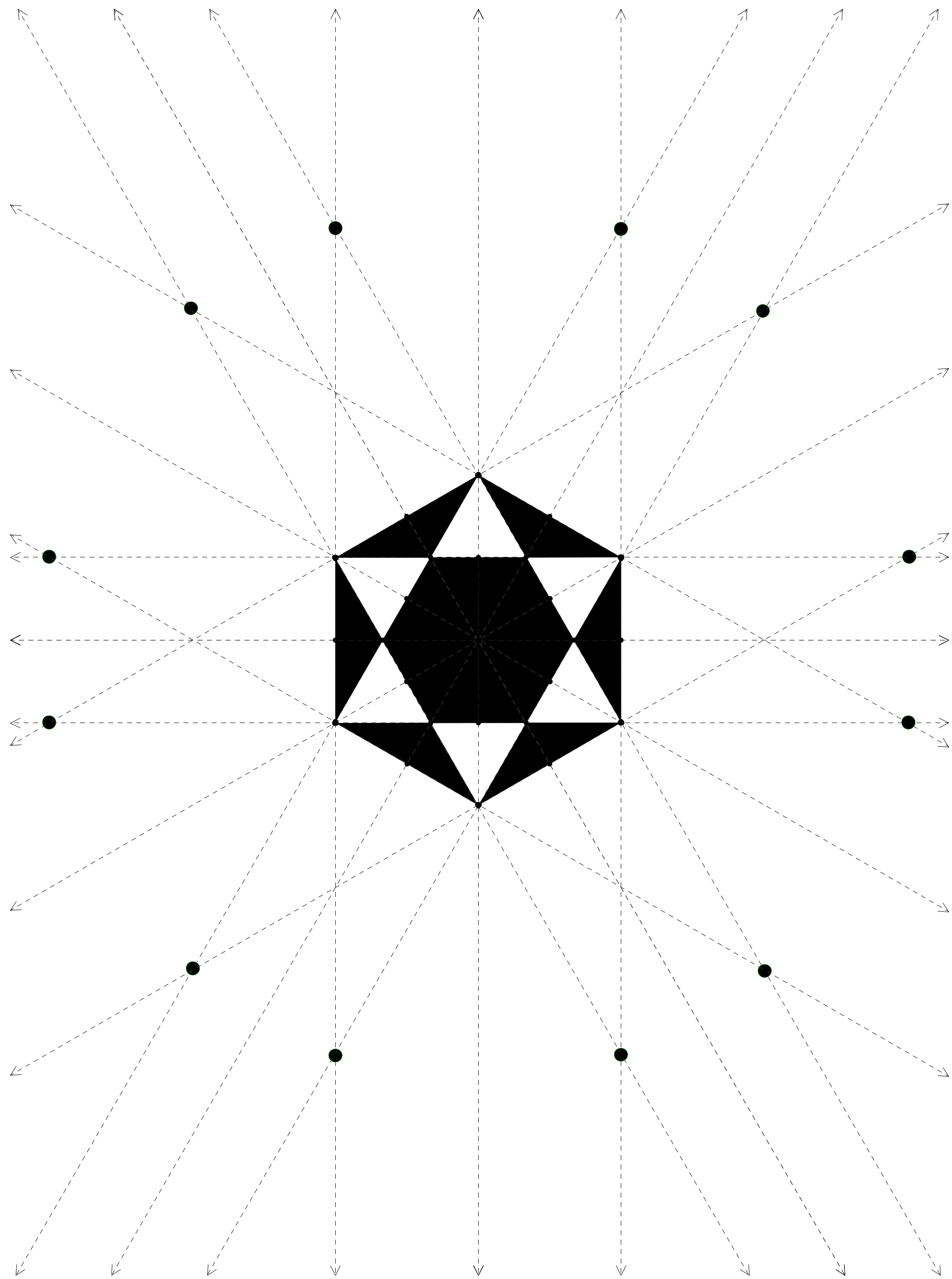

FIGURE 2. $SU(G_2)$; courtesy of C. Kriloff