UNITARIZABLE RAMIFIED PRINCIPAL SERIES OF SPLIT p-ADIC REDUCTIVE GROUPS

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ABSTRACT. These are notes for a talk given at AIM's "Atlas of Lie groups VI", July 2008. It is based on joint work with Dan Barbasch. The goal is to relate the unitarizability of subquotients of ramified p-adic principal series with the Iwahori-spherical unitary dual of endoscopic groups. The work on this project was supported by the NSF grant FRG-0554278.

1. NOTATION

Let $\mathbb{F}$ denote a p-adic field of characteristic 0. Later we will impose some restrictions on p (i.e., p not be a torsion prime for the group considered). Let $\mathbb{O}$ denote the ring of integers, and $\mathbb{F}^\times$, $\mathbb{O}^\times$ be the units. Let $\mathbb{F}_q$ denote the residue field.

We fix a linear algebraic connected reductive group G defined over $\overline{\mathbb{F}}$, corresponding to a root datum $(X, X^\vee, \Delta, \Delta^\vee)$. We assume that G is adjoint. Let $G(\mathbb{F})$ denote the split $\mathbb{F}$-form of G. Let $T(\mathbb{F}) = X^\vee \otimes_{\mathbb{Z}} \mathbb{F}^\times$ be the split torus, and $B(\mathbb{F}) \supset T(\mathbb{F})$ the $\mathbb{F}$-points of a Borel subgroup. We denote $K = G(\mathbb{O})$, a maximal compact open subgroup. Set ${}^0T = T(\mathbb{F}) \cap K \cong (\mathbb{O}^\times)^{\mathrm{rank}(G)}$.

Finally, let $G^\vee$ be the complex dual group, corresponding to the root datum $(X^\vee, X, \Delta^\vee, \Delta)$, $T^\vee = X \otimes_{\mathbb{Z}} (\mathbb{C}^\times)$, etc.

2. PARAMETERS FOR THE MINIMAL PRINCIPAL SERIES

Let $\chi : T(\mathbb{F}) \to \mathbb{C}$ be a character of the split torus. We denote the minimal principal series by

$$X(\chi) = \mathrm{Ind}_{B(\mathbb{F})}^{G(\mathbb{F})}(\chi \otimes 1). \tag{2.1}$$

We say that the character (or the principal series) is unramified if $\chi|_{{}^0T} = triv$. Otherwise, we say it is ramified, and we set ${}^0\chi = \chi|_{{}^0T} : {}^0T \to \mathbb{C}$.

Remark. Recall that in the real split case ${}^0T \cong (\mathbb{Z}^\times)^{\mathrm{rank}(G)} = (\mathbb{Z}/2)^{\mathrm{rank}(G)}$, and the familiar notation is M. The character ${}^0\chi$ is built from characters of $\mathbb{Z}/2$, so of *triv* and *sgn*. In the p-adic case, similarly, ${}^0\chi$ is built from smooth characters of $\mathbb{O}^\times$. Every such character is a lift from a unitary character of a finite group quotient of the form $\mathbb{O}^\times/1 + \varpi^m\mathbb{O}$, for some $m \in \mathbb{Z}_{\geq 0}$, where ϖ denotes a uniformizer of $\mathbb{F}$.

Let $\mathcal{W}_{\mathbb{F}}$ be the Weil group. Recall that the Galois group $Gal(\overline{\mathbb{F}}_q/\mathbb{F}_q) = \widehat{Z}$, i.e., the inverse limit of the quotients of $\mathbb{Z}$, and it is a topological closure of the cyclic group $\mathbb{Z}$ generated by a Frobenius element $Frob : x \mapsto x^q$ in $Gal(\overline{\mathbb{F}}_q/\mathbb{F}_q)$. The Galois group $\Gamma = Gal(\overline{\mathbb{F}}/\mathbb{F})$ has a short exact sequence

$$1 \longrightarrow I_{\mathbb{F}} \longrightarrow \Gamma \longrightarrow \widehat{Z} \longrightarrow 1, \tag{2.2}$$

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where $I_{\mathbb{F}} \subset \Gamma$ is the inertia group (compact). The Weil group $\mathcal{W}_{\mathbb{F}}$ is the subgroup of Γ which fits in the short exact sequence:

$$1 \longrightarrow I_{\mathbb{F}} \longrightarrow \mathcal{W}_{\mathbb{F}} \longrightarrow Z \longrightarrow 1, \tag{2.3}$$

and it is topologized by requiring $I_{\mathbb{F}}$ to be open in $\mathcal{W}_{\mathbb{F}}$. The Weil-Deligne group is

$$\mathcal{W}'_{\mathbb{F}} = \mathbb{C} \rtimes \mathcal{W}_{\mathbb{F}}, \tag{2.4}$$

where $w \in \mathcal{W}_{\mathbb{F}}$ acts on $\mathbb{C}$ by multiplication by $||w||$, the norm of w. In particular, a Frobenius $Frob$ acts by multiplication by q. The group $\mathcal{W}'_{\mathbb{F}}$ is locally compact via the product topology.

A Weil homomorphism $\phi : \mathcal{W}'_{\mathbb{F}} \to G^\vee$ is a continuous homomorphism satisfying certain properties. In particular, $\phi|_{\mathcal{W}_{\mathbb{F}}}$ should consist of semisimple elements and $\phi|_{\mathbb{C}}$ of unipotent elements. We say that ϕ is unramified if $\phi(I_{\mathbb{F}})$ consists of central elements of $G^\vee$.

Let $\mathcal{W}_{\mathbb{F}}^{ab}$ denote the abelian quotient of $\mathcal{W}_{\mathbb{F}}$. The Weil homomorphisms that parameterize minimal principal series descend to $\mathbb{C} \rtimes \mathcal{W}_{\mathbb{F}}^{ab}$. Let $I_{\mathbb{F}}^{ab}$ denote the image of $I_{\mathbb{F}}$ in $\mathcal{W}_{\mathbb{F}}^{ab}$. Recal that the reciprocity homomorphism $\tau_{\mathbb{F}}$ is an isomorphism of $\mathcal{W}_{\mathbb{F}}^{ab}$ onto $\mathbb{F}^\times$, and induces an isomorphism of $I_{\mathbb{F}}^{ab}$ onto $\mathbb{O}^\times$.

We fix for the rest of the paper a character ${}^0\chi : {}^0T \to \mathbb{C}^\times$. To it we attach a homomorphism

$$\widehat{{}^0\chi} : I_{\mathbb{F}}^{ab} \to T^\vee \subset G^\vee \tag{2.5}$$

as follows. The lattice $X^\vee$ in the root datum can be identified with $\mathrm{Hom}(\mathbb{F}^\times, T(\mathbb{F}))$, but also with $\mathrm{Hom}(T^\vee, \mathbb{C}^\times)$. Then $\widehat{{}^0\chi}$ is the unique homomorphism which makes the following diagram commutative for any $\lambda \in X^\vee$:

$$\begin{array}{ccccc} I_{\mathbb{F}}^{ab} & & \overset{\widehat{{}^0\chi}}{\dashrightarrow} & & T^\vee \\ \downarrow \tau_{\mathbb{F}} & & & & \downarrow \lambda \\ \mathbb{O}^\times & & & & \mathbb{C}^\times \\ & \searrow^{\lambda} & & \nearrow^{{}^0\chi} & \\ & & {}^0T & & \end{array} \tag{2.6}$$

Now we consider

$$\phi : \mathbb{C} \rtimes \mathcal{W}_{\mathbb{F}}^{ab} \to G^\vee \text{ such that } \phi|_{I_{\mathbb{F}}^{ab}} = \widehat{{}^0\chi}. \tag{2.7}$$

such that $\phi|_{I_{\mathbb{F}}^{ab}} = \widehat{{}^0\chi}$. Such homomorphisms parameterize L-packets which have subquotients of minimal principal series $X(\chi)$, with $\chi|_{{}^0T} = {}^0\chi$. Define

$$G^\vee(\widehat{{}^0\chi}) = \text{ centralizer in } G^\vee \text{ of } Im(\widehat{{}^0\chi}). \tag{2.8}$$

Lemma ([Ro98])**.** *$G^\vee(\widehat{{}^0\chi})$ is the centralizer of a single semisimple element in $G^\vee$.*

Since we assumed that G is adjoint, $G^\vee$ is simply connected, and by a theorem of Steinberg (with the necessary restrictions on p), $G^\vee(\widehat{{}^0\chi})$ is a connected reductive algebraic group. Let $G({}^0\chi)$ denote the split p-adic group whose dual is $G^\vee(\widehat{{}^0\chi})$.

Any parameter ϕ as in (2.7) has the image in fact in $G^\vee(\widehat{{}^0\chi})$. Denote by ϕ' the homomorphism obtained by restricting the range of ϕ to $G^\vee(\widehat{{}^0\chi})$. Then ϕ' is an unramified parameter for $G({}^0\chi)$.

This correspondence gives a bijection between (parameters) of parameters ϕ for the ${}^0\chi$-ramified principal series of $G(\mathbb{F})$ and the parameters ϕ' of the unramifies principal series of $G({}^0\chi)$. In other words, we have a correspondence of L-packets. In [Re02], this correspondence is refined by considering the component groups. Let $A_{G^\vee}(\phi)$ and $A_{G^\vee(\widehat{{}^0\chi})}(\phi')$ denote the component groups of the centralizers of the images of ϕ and ϕ' respectively. Let $\mathcal{B}^{\phi}_{G^\vee(\widehat{{}^0\chi})}$ and $\mathcal{B}^{\phi'}_{G^\vee(\widehat{{}^0\chi})}$ denote the varieties of Borel subgroups fixed by the images of ϕ and ϕ' respectively. We say that a representation of the component group is of Springer type if it appears in the natural action on the Borel-Moore homology of these Borel varieties. Then there is a natural isomorphism $A_{G^\vee}(\phi) \cong A_{G^\vee(\widehat{{}^0\chi})}(\phi')$, which induces a bijection Ψ of the component group representations of Springer type.

For simplicity, in this talk, we restrict to the case when the parameters are "real". For the general case, one needs to apply in addition the reduction

Definition. *We say that a homomorphism ϕ (or ϕ') is real if $\phi(Frob)$ is a hyperbolic (i.e, has positive real eigenvalues) semisimple element of $G^\vee$.*

3. Result

The main result is the following.

Theorem (Barbasch-Ciubotaru)**.** *The correspondence Ψ restricted to real parameters (in the sense of definition 2) takes hermitian representations to hermitian and unitary representations to unitary.*

We rephrase this in terms of subquotients of principal series. Let $s = \phi(Frob) \in T^\vee$ be a hyperbolic semisimple element. As it is well-known, s gives rise to a real unramified character of $T(\mathbb{F})$. Let $X_G({}^0\chi, s)$ denote the ramified minimal principal series induced from $\chi ={}^0 \chi \otimes s : T(\mathbb{F}) \to \mathbb{C}^\times$, where we regard $T(\mathbb{F}) ={}^0 T \times X^\vee$, by making a (bad!) choice of a uniformizer. Let $X_{G({}^0\chi)}(s)$ denote the unramified principal series for the p-adic group $G({}^0\chi)$. Then the correspondence Ψ can be regarded as a bijection

$$\Psi : \{\text{subquotients of } X_G({}^0\chi, s)\} \longleftrightarrow \{\text{subquotients of } X_{G({}^0\chi)}(s)\}. \tag{3.1}$$

In this bijection, the Langlands subquotient of $X_G({}^0\chi, s)$ corresponds to the spherical Langlands subquotient of $X_{G({}^0\chi)}(s)$. The claim of the theorem is that Ψ is also a bijection of hermitian modules, and of unitary modules.

Next, we give some indication of the machinery that goes into the proof. The idea is to use the Hecke algebra isomorphisms of [Ro98], and the techniques of [BM89].

3.1. The notation in this subsection is independent of the rest of the notes. Let H denote a complex connected reductive algebraic group with based root datum $(X^*, X_*, R, R^\vee, \Pi)$, ($\Pi$ denote the simple roots), pairing $\langle\ ,\ \rangle$, and (finite) Weyl group (W, S), where S is the set of simple reflections in the roots of Π. (Note that X^* is the lattice of characters of a maximal torus.) We let H^{sc} denote the simply connected cover of H.

Definition. *The affine (Iwahori-)Hecke algebra $\mathcal{H}(H)$ attached to H is the associative algebra generated over $\mathbb{C}[q, q^{-1}]$[1] by $\{T_w : w \in W\}$ and $\{\theta_\lambda, \lambda \in X^*\}$ subject*

[1] q can be considered as an indeterminate here, for us it is the cardinality of the residue field. In general, it can be specialized to any complex number, not a root of unity.

to the relations

$$T_wT_{w'} = T_{ww'},\ w, w' \in W,\ \textit{such that}\ \ell(ww') = \ell(w) + \ell(w'); \tag{3.2}$$

$$(T_{s_\alpha} + 1)(T_{s_\alpha} - q) = 0,\ s_\alpha \in S; \tag{3.3}$$

$$\theta_\lambda\theta_{\lambda'} = \theta_{\lambda+\lambda'},\ \lambda, \lambda' \in X^*; \tag{3.4}$$

$$\theta_\lambda T_{s_\alpha} - T_{s_\alpha}\theta_{s_\alpha(\lambda)} = (q-1)\frac{\theta_\lambda - \theta_{s_\alpha(\lambda)}}{1-\theta_{-\alpha}},\ s_\alpha \in S, \lambda \in X^*. \tag{3.5}$$

This is the Bernstein presentation of $\mathcal{H}(H)$. Let $\mathcal{H}_W(H)$ denote the subalgebra of $\mathcal{H}(H)$ generated by $\{T_w : w \in W\}$. This is known to be isomorphic to $\mathbb{C}[W]$. Also, let $\mathcal{A}(H)$ be the subalgebra of $\mathcal{H}(H)$ generated by $\{\theta_\lambda, \lambda \in X^*\}$; this is isomorphic to $\mathbb{C}[X^*]$. Similarly we define $\mathcal{H}(H^{sc})$.

Note that in the nontrivial commutation relation, the quotient is a priori only defined over the field of quotients of $\mathcal{A}(H)$. It is in $\mathcal{A}(H)$ however since $\theta_\lambda - \theta_{s_\alpha(\lambda)} = \theta_\lambda - \theta_{\lambda - \langle\lambda,\check\alpha\rangle\alpha} = \theta_\lambda(1 - \theta_{-\alpha}^{\langle\lambda,\check\alpha\rangle})$.

We also recall that the center of $\mathcal{H}(H)$ is $\mathcal{A}(H)^W$, i.e, the W-invariants in $\mathbb{C}[X^*]$.

Example 1. Assume $H = SL(2)$. Let T correspond to the nontrivial element s_α of W, and θ to the generator of X^*, such that $\theta^2 = \theta_\alpha$. Then $\mathcal{H}(SL(2,\mathbb{C}))$ is generated by T and θ subject to

$$(T+1)(T-q) = 0; \tag{3.6}$$

$$\theta T - T\theta^{-1} = (q-1)\theta. \tag{3.7}$$

The center is generated by $\theta^2 + \theta^{-2}$. In terms of the Iwahori-Matsumoto affine generators, the presentation is

$$(T_1+1)(T_1-q) = 0; \tag{3.8}$$

$$(T_2+1)(T_2-q) = 0; \tag{3.9}$$

$$\tau T_1 = T_2\tau. \tag{3.10}$$

Note the presence of the generator τ which acts by the outer automorphism of the affine Dynkin diagram of type $\widetilde{A}_1$ (T_1, T_2 correspond to the two affine simple roots). This algebra has only two one-dimensional representations: the Steinberg module $T_1 = T_2 = -1$, and the trivial $T_1 = T_2 = q$.

Example 2. Assume $H = PGL(2)$. Let T correspond to the nontrivial element s_α of W, and θ' to the generator of X^*, such that $\theta' = \theta_\alpha$. Then $\mathcal{H}(PGL(2,\mathbb{C}))$ is generated by T and θ' subject to

$$(T+1)(T-q) = 0; \tag{3.11}$$

$$\theta' T - T(\theta')^{-1} = (q-1)(1+\theta'). \tag{3.12}$$

The center is generated by $\theta' + (\theta')^{-1}$. In terms of the Iwahori-Matsumoto affine generators, the presentation is

$$(T_1+1)(T_1-q) = 0; \tag{3.13}$$

$$(T_2+1)(T_2-q) = 0; \tag{3.14}$$

Again T_1, T_2 correspond to the two affine simple roots. In addition to the Steinberg and trivial modules, this algebra has two more one-dimensional representations: $T_1 = -1, T_2 = q$, and $T_1 = q, T_2 = -1$. This is related by the Borel-Casselman correspondence to the fact that the unitary spherical minimal principal series of

$SL(2,\mathbb{Q}_p)$ is reducible for one unramified (purely imaginary) character (and the two summands correspond to these two "extra" modules), but it is irreducible for $PGL(2,\mathbb{Q}_p)$.

There is an injection $\mathcal{H}(PGL(2,\mathbb{C})) \hookrightarrow \mathcal{H}(SL(2,\mathbb{C}))$ given by $T \mapsto T$, and $\theta' \mapsto \theta^2$ which identifies the centers. This is a general fact, the isogeny of root data induces on injection

$$\mathcal{H}(H) \hookrightarrow \mathcal{H}(H^{sc}). \tag{3.15}$$

In [KL87], the irreducible modules of $\mathcal{H}(H^{sc})$ are classified. Using [KL87] and results of Ram-Rammage, [Re02] gives a classification of irreducibles for $\mathcal{H}(H)$ by restriction via (3.15). We should mention that the same classification can be deduced via [Lu89]'s reduction to the affine graded Hecke algebra (where the isogenies disappear).

Recall that in [KL87] the irreducible modules of $\mathcal{H}(H^{sc})$ are classified by H^{sc}-conjugacy classes of triples (s, e, ψ), where $s \in H^{sc}$ is semisimple, $u \in H^{sc}$ is unipotent, such that $Ad(s)u = qu$, and ψ is a representation of the component group $A_{s,u}$ of the mutual centralizer of s, u in H^{sc} such that ψ appears in the natural action of $A_{s,u}$ on the Borel-Moore homology $H_*(\mathcal{B}_{s,u})$. Here $\mathcal{B}_{s,u}$ is the variety of Borel subgroups on H^{sc} containing both s and u.

Lemma ([Re02],[Lu89])**.** *Assume $s \in H^{sc}$ is hyperbolic. Then the irreducible module $V_{s,u,\psi}$ of $\mathcal{H}(H^{sc})$ is irreducible when restricted to $\mathcal{H}(H)$.*

In [Re02], certain R-groups which control this restriction are computed. They are subgroups of the group of components of the centralizer of s in H. But since s is hyperbolic, the centralizer of s in H is connected. As mentioned before, one can also use the restriction to the graded Hecke algebra $\mathbb{H}$. By a theorem in [Lu89], if s is hyperbolic, the irreducible modules of $\mathcal{H}(H)$ or $\mathcal{H}(H^{sc})$ are irreducible $\mathbb{H}$-modules. The conclusion follows.

Theorem ([KL87])**.** *An irreducible $\mathcal{H}(H^{sc})$-module $V_{s,e,\psi}$ with s hyperbolic, is tempered if there exists a homomorphism $\phi : sl(2,\mathbb{C}) \to \mathfrak{h}$ such that*

$$\phi\begin{pmatrix} \frac{\log q}{2} & 0 \\ 0 & \frac{\log q}{2} \end{pmatrix} = \log s, \ \textit{and } \phi\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \log u. \tag{3.16}$$

Recall that the tempered modules for an affine Hecke algebra are defined in terms of Casselman's criterion. For us, the W-structure of real tempered modules is important.

Proposition ([BM89])**.** *There exists 1-1 correspondence*

$$\{\textit{real tempered } \mathcal{H}(H^{sc})\textit{-modules}\} \longleftrightarrow \widehat{W}. \tag{3.17}$$

If one regards both sides as bases for the Grothendieck group of W, the change of basis is upper uni-triangular, and in particular the set of real tempered modules are linearly independent over W.

This is obtained by using the realization of the restriction of tempered modules to $\mathcal{H}_W$, the Springer correpondence and results of Borho-MacPherson. In fact, equation (3.17) is nothing else by Springer's correspondence.

Combining the results so far, one concludes:

Corollary. *The same result as in proposition 3.1 holds with H in place of H^{sc}.*

3.2. Let $\mathcal{R}(G(\mathbb{F}))$ denote the category of smooth complex representations of $G(\mathbb{F})$. It admits a Bernstein decomposition

$$\mathcal{R}(G(\mathbb{F})) = \prod_{\mathfrak{s}} R_{\mathfrak{s}}(G(\mathbb{F})), \tag{3.18}$$

where the irreducibles in each such subcategory have the same supercuspidal support modulo unramified twists. For example, the character ${}^0\chi : {}^0T \to \mathbb{C}^\times$ defines a subcategory $\mathcal{R}_{{}^0\chi}(G(\mathbb{F}))$, where the objects are (finite length) smooth representations whose every composition factor appears as a subquotient of a minimal principal series $X(\chi)$, for some character χ of $T(\mathbb{F})$, whose restriction to ${}^0T$ is ${}^0\chi$.

Lemma (Casselman)**.** *If $\overline{\pi} \in R_{\mathfrak{s}}(G(\mathbb{F}))$ is an irreducible Langlands quotient of a standard representation π, then $\pi \in R_{\mathfrak{s}}(G(\mathbb{F}))$.*

As an example of Bushnell-Kutzko's theory of types, generalizing the well-known Borel-Casselman correspondence, [Ro98] constructs a pair $(J_{{}^0\chi}, \widetilde{{}^0\chi})$, where ${}^0T \subset J_{{}^0\chi} \subset K$ is a parahoric subgroup, and $\widetilde{{}^0\chi} : J_{{}^0\chi} \to \mathbb{C}^\times$ extends ${}^0\chi$.

Example. If $G(\mathbb{F}) = SL(2, \mathbb{Q}_p)$, and ${}^0\chi : {}^0T \to \mathbb{C}^\times$ has conductor $m \geq 0$ (i.e., it factors through $1 + p^{m+1}\mathbb{Z}_p$, but not through $1 + p^m\mathbb{Z}_p$), then

$$J_{{}^0\chi} = \begin{pmatrix} \mathbb{Z}_p^\times & p^{[\frac{m-1}{2}]}\mathbb{Z}_p \\ p^{[\frac{m}{2}]}\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}. \tag{3.19}$$

In particular, if ${}^0\chi$ has conductor 0, then $J_{{}^0\chi}$ is an Iwahori subgroup.

Let $Rep(G(\mathbb{F}), J_{{}^0\chi}, \widetilde{{}^0\chi})$ denote the category of smooth representations π of $G(\mathbb{F})$ such that π is generated by

$$\pi^{\widetilde{{}^0\chi}} := \mathrm{Hom}_{J_{{}^0\chi}}[\pi : \widetilde{{}^0\chi}]. \tag{3.20}$$

Firstly, one proves that:

Theorem ([Ro98])**.** *$R_{{}^0\chi}(G(\mathbb{F})) = Rep(G, , J_{{}^0\chi}, \widetilde{{}^0\chi})$.*

In particular, $Rep(G, J_{{}^0\chi}, \widetilde{{}^0\chi})$ is closed under taking subquotients. If $\mathcal{H}(G(\mathbb{F}), J_{{}^0\chi}, \widetilde{{}^0\chi})$ denotes the Hecke algebra (with convolution) of locally constant functions

$$g : G(\mathbb{F}) \to \mathbb{C}, \text{ such that } g(j_1 x j_2) = \widetilde{{}^0\chi}(j_1)^{-1} g(x) \widetilde{{}^0\chi}(j_2)^{-1}, \ j_1, j_2 \in J_{{}^0\chi}, \tag{3.21}$$

then one finds:

Theorem ([Ro98])**.** *The functor $\pi \mapsto \pi^{\widetilde{{}^0\chi}}$ is an equivalence of categories between $Rep(G, J_{{}^0\chi}, \widetilde{{}^0\chi})$ and $\mathcal{H}(G(\mathbb{F}), J_{{}^0\chi}, \widetilde{{}^0\chi})$.*

Finally one needs to determine the structure of $\mathcal{H}(G(\mathbb{F}), J_{{}^0\chi}$. It is important for unitarity that these Hecke algebras have a natural $*$-operation given on functions by $g \mapsto g^*$, where $g^*(x) = \overline{g(x^{-1})}$.

Theorem ([Ro98])**.** *There is a $*$-preserving, support preserving isomorphism*

$$\mathcal{H}(G(\mathbb{F}), J_{{}^0\chi}, \widetilde{{}^0\chi}) \cong \mathcal{H}(G^\vee(\widehat{{}^0\chi})), \tag{3.22}$$

where the second algebra is the Iwahori-Hecke algebra for the split p-adic group $G({}^0\chi)$. Moreover, this isomorphism preserves the Bernstein decompositions, i.e.,

1. $\mathcal{H}(K, J_{{}^0\chi}, \widetilde{{}^0\chi}) \cong \mathcal{H}_{W(\widehat{{}^0\chi})}(G^\vee(\widehat{{}^0\chi}))$, *and*
2. $\mathcal{H}(T, {}^0T, {}^0\chi) \cong \mathbb{C}[X^\vee]$.

The statement about the "W-parts" of the Hecke algebras is not explicit in [Ro98]. It follows however by examining the base case (level 0, due to Morris) and using the preservation of support for the other Hecke algebra isomorphisms in [Ro98] which reduce the problem to level 0. Consequently, for all our purposes, we can identify the two Hecke algebras.

Finally note that the category of $\mathcal{H}(G^\vee(\widehat{^0\chi}))$-modules is isomorphic to the Iwahori-spherical category of $G(^0\chi)$ by the Borel-Casselman correspondence. The irreducible objects in the latter are precisely the subquotients of unramified principal series.

3.3. We conclude by applying the [BM89] methods to this setting. Using the $*$-operations, one defines hermitian and unitary modules for the Hecke algebras. We restrict to "real" representations in the sense of definition 2. The methods are p-adic analogues of the fundamental definitions and formulas of [Vo84].

If π is a smooth admissible $G(\mathbb{F})$-representation, let $\theta_K(\pi)$ denote the K-character of π:

$$\theta(\pi) = \sum_{\mu\in\widehat{K}} m_\pi(\mu)\mu, \tag{3.23}$$

where $m_\pi(\mu)$ is the (finite) multiplicity of μ in π. Similarly define for a module π' of a Hecke algebra $\mathcal{H}$ of the type appearing so far, $\theta_W(\pi')$, the $\mathcal{H}_w$-character.

If π admits a hermitian form, then define the formal signature character:

$$\Sigma(\pi) = (\sum_{\mu\in\widehat{K}} p(\mu)\mu, \sum_{\mu\in\widehat{K}} q(\mu)\mu)), \tag{3.24}$$

where $p(\mu), q(\mu)$ are the number of positive and negative eigenvalues respectively, of the nondegenerate hermitian form on μ obtained from the one of π. So $p(\mu)+q(\mu) = m_\pi(\mu)$. One can define the same notion for the Hecke algebra.

Lemma. *If π is a real irreducible representation in $R_{^0\chi}(G(\mathbb{F}))$, then there exist real tempered modules $\pi_1, \dots, \pi_n$ in $R_{^0\chi}(G(\mathbb{F}))$ and integers $a_1, \dots, a_n$, such that*

$$\theta_K(\pi) = \sum_{i=1}^{n} a_i\theta_K(\pi_i). \tag{3.25}$$

For the proof one needs to use some induction on the length of the Langlands parameter.

Theorem. *The signature character of real hermitian $\pi \in R_{^0\chi}(G(\mathbb{F}))$ can be written as*

$$\Sigma(\pi) = (\sum_{i=1}^{n} a_i\theta_K(\pi_i), \sum_{i=1}^{n} b_i\theta_K(\pi_i)), \tag{3.26}$$

for some real tempered $\pi_i \in R_{^0\chi}(G(\mathbb{F}))$ and integers a_i, b_i.

We can write the analogous theorem in the category of $\mathcal{H}(G^\vee(\widehat{^0\chi}))$-modules. We need to relate the tempered modules in the two categories via the correspondences of section 3.2.

Proposition. *The correspondence $\pi \mapsto \pi^{\widetilde{^0\chi}}$ between the categories $R_{^0\chi}(G(\mathbb{F}))$ and $\mathcal{H}(G^\vee(\widehat{^0\chi}))$-modules takes tempered modules to tempered modules.*

Finally, we can put all the pieces together.

Theorem. *The correspondence* $\pi \mapsto \pi^{\widetilde{{}^0\chi}}$ *between the categories* $R_{{}^0\chi}(G(\mathbb{F}))$ *and* $\mathcal{H}(G^\vee(\widehat{{}^0\chi}))$*-modules takes real hermitian irreducible modules to real hermitian irreducible modules, and real unitary irreducible modules to real unitary irreducible modules.*

The only nontrivial part is that $\pi^{\widetilde{{}^0\chi}}$ unitary implies π unitary. The discussion so far implies that if $\Sigma(\pi) = (\sum_{i=1}^n a_i\theta_K(\pi_i), \sum_{i=1}^n b_i\theta_K(\pi_i))$ is the signature character of π, then $\Sigma(\pi^{\widetilde{{}^0\chi}}) = (\sum_{i=1}^n a_i\theta_W(\pi_i^{\widetilde{{}^0\chi}}), \sum_{i=1}^n b_i\theta_W(\pi_i^{\widetilde{{}^0\chi}}))$ is the signature character of $\pi^{\widetilde{{}^0\chi}}$. If $\pi^{\widetilde{{}^0\chi}}$ is unitary, then we must have

$$\sum_{i=1}^n b_i\theta_W(\pi_i^{\widetilde{{}^0\chi}})) = 0. \tag{3.27}$$

But then $b_i = 0$ for all $i = 1, n$ since $\theta_W(\pi_i^{\widetilde{{}^0\chi}})$ are linearly W-independent as explained in section 3.1. So the form on π is positive definite.

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