

You should complete this assignment on your own. You are not allowed to collaborate with others, and you are only allowed to use the textbook (Jacobson) and the class notes. No other resources (books, internet, friends) are allowed. It is OK to turn in handwritten solutions.

1. (a) Let G be a group and H a subgroup of finite index. Show that there exists a normal subgroup N of G contained in H and also of finite index.
(b) Let G be a group and let H_1, H_2 be subgroups of finite index. Prove that $H_1 \cap H_2$ has finite index.
2. Let D be the set of complex numbers of the form $m + n\sqrt{-3}$ such that either both m, n are integers, or both m, n are halves of odd integers. Show that:
 - (a) D is a subring of \mathbb{C} ;
 - (b) D is a Euclidean domain.
3. Let R be a factorial domain with field of fractions K . Let $f(x)$ be a polynomial in $R[x]$. Assume that $f(x)$ can be factored into polynomials of lower degree in $K[x]$. Show that $f(x)$ can be factored into polynomials of lower degree in $R[x]$.
4. Let R be a noncommutative ring. As in class, a proper ideal is called maximal if it is not contained in any proper ideal. An ideal is called prime if whenever $ab \in R$, we have $a \in R$ or $b \in R$. We showed in class that the following statements are true when R is commutative. Prove or disprove them when R is noncommutative:
 - (a) If R/M is a field then M is maximal.
 - (b) If R/M is a division ring, then M is maximal.
 - (c) If M is maximal, then R/M is a division ring.
 - (d) If R/P is an integral domain, then P is prime.
 - (e) If P is prime, then R/P is an integral domain.
5. Let $I \subset \mathbb{Z}[x]$ be the ideal consisting of all polynomials $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ such that 8 divides a_0 , 4 divides a_1 , and 2 divides all the other coefficients. Find a finite set of generators of I .
6. Let N be a normal subgroup of a finite group G . Assume that the center of N is trivial and that any automorphism of N is inner. Show that there exists a normal subgroup H of G such that $G \cong N \times H$. Can you give a nontrivial example of such a group N ?