

1. Let A be a ring and I an ideal with the property that

$$\bigcap_{\nu=1}^{\infty} I^{\nu} = \{0\}.$$

Then one can define notions of convergent and Cauchy sequences in the I -adic topology as follows. A sequence $\{a_n\} \subset A$ is called convergent to $a \in A$ if given ν there exists an integer N such that $a_n - a \in I^{\nu}$ for all $n \geq N$. Similarly, $\{a_n\}$ is Cauchy if given ν there exists N such that $a_n - a_m \in I^{\nu}$ for all $n, m \geq N$. The ring A is called complete if every Cauchy sequence converges.

- (a) Let R be a discrete valuation local ring with maximal ideal \mathfrak{m} . Verify that one can define the notion of \mathfrak{m} -adic topology.
- (b) In the last homework we saw two examples of discrete valuation rings. Check if they are complete in their \mathfrak{m} -adic topologies.
2. Recall the elementary symmetric polynomials s_i , $i = 1, \dots, n$, in $\mathbb{Z}[x_1, \dots, x_n]$. Let p_i be the sum of i powers of the variables, i.e., $p_i = x_1^i + \dots + x_n^i$. Prove Newton's formulas:

$$\begin{aligned} p_1 - s_1 &= 0 \\ p_2 - s_1 p_1 + 2s_2 &= 0 \\ p_3 - s_1 p_2 + s_2 p_1 - 3s_3 &= 0 \\ \dots & \\ p_i - s_1 p_{i-1} + s_2 p_{i-2} - \dots + (-1)^{i-1} s_{i-1} p_1 + (-1)^i i s_i &= 0. \end{aligned}$$

Deduce that $\{p_i\}$ are also generators for the ring of symmetric polynomials, but over \mathbb{Q} , not over \mathbb{Z} .

3. (a) Show that the polynomials $x^4 + 1$ and $x^6 + x^3 + 1$ are irreducible over \mathbb{Q} .
- (b) Show that a polynomial of degree 3 over a field is either irreducible or has a root in the field. Is $x^3 - 5x^2 + 1$ irreducible over \mathbb{Q} ?
- (c) Show that the polynomial in two variables $x^2 + y^2 - 1$ is irreducible over \mathbb{Q} . Is it irreducible over \mathbb{C} ?
4. Determine all maximal ideals of the ring $\mathbb{Z}[x]/(120, x^3 + 1)$.