Math 6310, Assignment 4

Due Monday, October 29 in class.

1. Let $F$ be a field. A discrete valuation on $F$ is a map $\nu : F^\times \rightarrow \mathbb{Z}$ such that
   
   (a) $\nu(ab) = \nu(a) + \nu(b)$;
   (b) $\nu$ is surjective;
   (c) $\nu(x + y) \geq \min(\nu(x), \nu(y))$.

   The valuation ring of $\nu$ is defined to be $R = \{ x \in F^\times | \nu(x) \geq 0 \} \cup \{0\}$.

   (a) Prove that $R$ is a subring of $F$ that contains the identity.
   (b) Prove that for every nonzero element $x \in F$, either $x$ or $x^{-1}$ is in $R$.
   (c) Prove that an element $x$ is a unit of $R$ if and only if $\nu(x) = 0$.
   (d) Prove that $R$ is a local ring (i.e., has a unique maximal ideal).

2. Let $F = \mathbb{Q}$ and fix a prime $p$. Define $\nu : \mathbb{Q}^\times \rightarrow \mathbb{Z}$ as follows. If $x = \frac{a}{b} \in \mathbb{Q}^\times$, write $x = p^\ell a' \frac{1}{b'}$, where $p$ does not divide $a'$ and $b'$. (In other words, you factor out all the powers of $p$ and lump them in the $p^\ell$ factor.) Then define $\nu(x) = \ell$.

   (a) Describe the corresponding valuation ring $R$.
   (b) Compute the units in $R$ in this example.

3. Let $F$ be a field and consider formal Laurent power series over $F$:

   $$F((x)) := \left\{ \sum_{i \geq n} a_i x^i | a_i \in F \text{ and } n \in \mathbb{Z} \right\}.$$

   (a) Define natural operations of addition and multiplication on $F((x))$ and prove that $F((x))$ is a field.
   (b) Define $\nu : F((x))^\times \rightarrow \mathbb{Z}$ by defining $\nu$ of a formal Laurent series $a_n x^n + a_{n+1} x^{n+1} + a_{n+2} x^{n+2} + \ldots$ (with $a_n \neq 0$) to be $n$. Prove that $\nu$ is a valuation in the sense of the previous problem.
   (c) Show that the valuation ring of $\nu$ is the subring of $F((x))$ of formal Laurent series in which no negative powers of $x$ occur. (This subring is called the ring of formal power series in $x$, typically denoted $F[[x]]$.)
4. Let $A$ be a ring and $I$ an ideal with the property that
\[ \cap_{\nu=1}^{\infty} I^\nu = \{0\}. \]
Then one can define notions of convergent and Cauchy sequences in the $I$-adic topology as follows. A sequence $\{a_n\} \subset A$ is called convergent to $a \in A$ if given $\nu$ there exists an integer $N$ such that $a_n - a \in I^\nu$ for all $n \geq N$. Similarly, $\{a_n\}$ is Cauchy if given $\nu$ there exists $N$ such that $a_n - a_m \in I^\nu$ for all $n, m \geq N$. The ring $A$ is called complete if every Cauchy sequence converges.

(a) Let $R$ be a discrete valuation local ring with maximal ideal $m$. Verify that $m$ satisfies (1).

(b) Check if the two discrete valuation rings defined above are complete in their $m$-adic topologies.