

## Math 6310, Assignment 3

Due by Friday, October 7 at 5:00pm (in class or my office LCB 106)

Assume that all the rings below have identity  $1 \neq 0$ .

1. Let  $H$  be a subgroup of a finite abelian group  $G$ . Show that  $G$  has a subgroup that is isomorphic to  $G/H$ .
2. Let  $G$  be a group and let  $\mathbf{Set}(G)$  be the category of sets with a  $G$ -action. Let  $F : \mathbf{Set}(G) \rightarrow \mathbf{Set}$  be the forgetful functor which assigns to each  $G$ -set the set itself. Show that  $\text{Aut}(F)$  is naturally isomorphic to  $G$ .
3. Let  $R$  be a ring. Define the nilradical of  $R$  to consist of all elements  $x \in R$  such that  $x^n = 0$  for some  $n \in \mathbb{Z}$ .
  - (a) Suppose  $R$  is commutative. Prove that the nilradical is an ideal of  $R$ .
  - (b) Is the nilradical an ideal even if  $R$  is noncommutative?
4. Let  $R$  be an integral domain which is not a field. Show that  $R[x]$  is not a principal ideal domain.
5. Let  $I$  be an ideal in  $R$  and let  $M_n(I)$  denote the set of  $n \times n$  matrices with entries in  $I$ .
  - (a) Show that  $M_n(I)$  is an ideal in  $M_n(R)$ .
  - (b) Prove that every ideal in  $M_n(R)$  has the form  $M_n(I)$  for some ideal  $I$  of  $R$ , and that the map  $I \rightarrow M_n(I)$  is a bijective map of the set of ideals of  $R$  onto the set of ideals of  $M_n(R)$ .
6. Let  $F$  be a field. A discrete valuation on  $F$  is a map  $\nu : F^\times \rightarrow \mathbb{Z}$  such that
  - (a)  $\nu(ab) = \nu(a) + \nu(b)$ ;
  - (b)  $\nu$  is surjective;
  - (c)  $\nu(x + y) \geq \min(\nu(x), \nu(y))$ .

The valuation ring of  $\nu$  is defined to be  $R = \{x \in F^\times \mid \nu(x) \geq 0\} \cup \{0\}$ .

- (a) Prove that  $R$  is a subring of  $F$  that contains the identity.

- (b) Prove that for every nonzero element  $x \in F$ , either  $x$  or  $x^{-1}$  is in  $R$ .
- (c) Prove that an element  $x$  is a unit of  $R$  if and only if  $\nu(x) = 0$ .
- (d) Prove that  $R$  is a local ring (i.e., has a unique maximal ideal).
7. Let  $F = \mathbb{Q}$  and fix a prime  $p$ . Define  $\nu : \mathbb{Q}^\times \rightarrow \mathbb{Z}$  as follows. If  $x = \frac{a}{b} \in \mathbb{Q}^\times$ , write  $x = p^\ell \frac{a'}{b'}$ , where  $p$  does not divide  $a'$  and  $b'$ . (In other words, you factor out all the powers of  $p$  and lump them in the  $p^\ell$  factor.) Then define  $\nu(x) = \ell$ .
- (a) Describe the corresponding valuation ring  $R$ .
- (b) Compute the units in  $R$  in this example.
8. Let  $F$  be a field and consider formal Laurent power series over  $F$ :

$$F((x)) := \left\{ \sum_{i \geq n} a_i x^i \mid a_i \in F \text{ and } n \in \mathbb{Z} \right\}.$$

- (a) Define natural operations of addition and multiplication on  $F((x))$  and prove that  $F((x))$  is a field.
- (b) Define  $\nu : F((x))^\times \rightarrow \mathbb{Z}$  by defining  $\nu$  of a formal Laurent series  $a_n x^n + a_{n+1} x^{n+1} + a_{n+2} x^{n+2} + \dots$  (with  $a_n \neq 0$ ) to be  $n$ . Prove that  $\nu$  is a valuation in the sense of the previous problem.
- (c) Show that the valuation ring of  $\nu$  is the subring of  $F((x))$  of formal Laurent series in which no negative powers of  $x$  occur. (This subring is called the ring of formal power series in  $x$ , typically denoted  $F[[x]]$ .)