Math 6310, Assignment 3. Due by Friday, October 5 (by email or my office).

1. Let $H$ be a subgroup of a finite abelian group $G$. Show that $G$ has a subgroup that is isomorphic to $G/H$. (Hint: use dual groups.)

2. Let $G$ be a group and let $\text{Set}(G)$ be the category of sets with a $G$-action. Let $F : \text{Set}(G) \longrightarrow \text{Set}$ be the forgetful functor which assigns to each $G$-set the set itself. Show that $\text{Aut}(F)$ is naturally isomorphic to $G$.

3. Let $N$ be a normal subgroup of a finite group $G$. Assume that the center of $N$ is trivial and that any automorphism of $N$ is inner.
   (a) Show that there exists a normal subgroup $H$ of $G$ such that $G \cong N \times H$.
   (b) Conclude that if $S_n$, $n \neq 6$ is a normal subgroup of a group $G$, then $G$ is a direct product of $S_n$ with some other group.
   (c) Can you give an example where the conclusion in (b) is false for $S_6$?

4. Prove that a presentation of the symmetric group $S_n$ is given by
   $$\langle s_1, \ldots, s_{n-1} | s_i^2 = 1, (s_is_{i+1})^3 = 1, (s_is_j)^2 = 1, \text{ if } |i - j| > 1 \rangle.$$ 

5. Let $p$ be a prime and $H(\mathbb{Z}/p\mathbb{Z})$ the Heisenberg group over $\mathbb{Z}/p\mathbb{Z}$ (see homework 1). Identify $H(\mathbb{Z}/p\mathbb{Z})$ as a semidirect product $(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}) \rtimes \mathbb{Z}/p\mathbb{Z}$. 

6. In $\text{GL}(n, \mathbb{C})$, let $B$ be the subgroup of upper triangular matrices and $N$ the subgroup of $B$ with 1’s on the diagonal. Prove that:
   (a) $N$ is normal in $B$.
   (b) $B$ is solvable and $N$ is nilpotent.

7. Let $R$ be a ring with 1. Define the nilradical of $R$ to consist of all elements $x \in R$ such that $x^n = 0$ for some $n \in \mathbb{Z}$.
   (a) Suppose $R$ is commutative. Prove that the nilradical is an ideal of $R$.
   (b) Is the nilradical an ideal even if $R$ is noncommutative?

8. Let $R$ be a commutative ring with 1. Let $I$ be an ideal in $R$ and let $M_n(I)$ denote the set of $n \times n$ matrices with entries in $I$.
   (a) Show that $M_n(I)$ is an ideal in $M_n(R)$. 

(b) Prove that every ideal in $M_n(R)$ has the form $M_n(I)$ for some ideal $I$ of $R$, and that the map $I \mapsto M_n(I)$ is a bijective map of the set of ideals of $R$ onto the set of ideals of $M_n(R)$.