Math 6310, Assignment 2 Due in class: Wednesday, September 21

1. (a) Show that there are no simple groups of order 56.

   (b) Show that any group of order 105 has a normal Sylow 5-subgroup and a normal Sylow 7-subgroup.

2. Compute the derived series for $S_4$ and $S_5$. Is $S_4$ solvable? What about $S_5$? (As mentioned in class, these examples have important historical meaning, and will appear again when we study Galois theory.)

3. Let $G$ be a group. Recall the group $\mathsf{Aut}(G)$ of automorphisms of $G$. For every $g \in G$, define $\varphi_g \in \mathsf{Aut}(G)$ given by $\varphi_g(x) = gxg^{-1}$, for all $x \in G$. Set $\mathsf{Inn}(G) = \{\varphi_g : \ g \in G\}$ (the group of inner automorphisms).

   (a) Prove that $\mathsf{Inn}(G)$ is a normal subgroup of $\mathsf{Aut}(G)$. Define $\mathsf{Out}(G) = \mathsf{Aut}(G)/\mathsf{Inn}(G)$ (the group of outer automorphisms).

   (b) Prove that if $\mathcal{K}$ is a conjugacy class in $G$, and $\varphi \in \mathsf{Aut}(G)$, then $\varphi(\mathcal{K})$ is a conjugacy class in $G$ too.

   The following questions refer to $S_n$.

   (c) Let $\mathcal{K}$ be the conjugacy class of transpositions in $S_n$, and let $\mathcal{K}'$ be the conjugacy class of any element of order 2 which is not a transposition. Prove that if $n \neq 6$, then $|\mathcal{K}'| \neq |\mathcal{K}|$. Conclude that any automorphism of $S_n$, $n \neq 6$ sends transpositions to transpositions.

   (d) Using the generators of $S_n$, $(1\ 2), (1\ 3), \ldots, (1\ n)$, deduce now that if $n \neq 6$, then $|\mathsf{Aut}(S_n)| \leqslant n!$. Conclude that $\mathsf{Aut}(S_n) = \mathsf{Inn}(S_n)$, and so $\mathsf{Out}(S_n) = \{1\}$ for $n \neq 6$.

   (e) Show that in $S_6$, the map

   $$(1\ 2) \longmapsto (1\ 2)(3\ 4)(5\ 6), \quad (2\ 3) \longmapsto (1\ 4)(2\ 5)(3\ 6), \quad (3\ 4) \longmapsto (1\ 3)(2\ 4)(5\ 6),$$
   $$(4\ 5) \longmapsto (1\ 2)(3\ 6)(4\ 5), \quad (5\ 6) \longmapsto (1\ 4)(2\ 3)(5\ 6),$$

   extends to an automorphism of $S_n$, which is not inner. This implies that $|\mathsf{Out}(S_6)| \geqslant 2$, in fact, it's not too hard now to show that $\mathsf{Out}(S_6) = C_2$, the cyclic 2-group.

4. Let $k$ denote the field $\mathbb{C}$ or $\mathbb{R}$. Let $\mathfrak{g} = sl(2, k)$ be the set of matrices of trace 0 with coefficients in $k$. Let $G = SL(2, k)$ be the group of invertible matrices of determinant 1 with coefficients in $k$.

(a) Verify that $G$ acts on $\mathfrak{g}$ by conjugation.

(b) Determine the orbits of this action in the two cases $k = \mathbb{C}$ and $k = \mathbb{R}$.

5. Let $G$ be a group of order $p^3$ and let $Z$ be the center of $G$. Let $C_p$ denote the cyclic group of order $p$. Show the if $G$ is not abelian, then $Z \cong C_p$ and $G/Z \cong C_p \times C_p$. (Using this result you can now determine rigorously, up to isomorphism, all the groups of order 8, see the next exercise.)

6. Classify all groups of order less than or equal to 10.

7. Let $G$ be a $p$-group. If $H$ is normal in $G$ and $H$ has order $p$, show that $H$ must be central in $G$ (i.e., a subgroup of the center of $G$).

8. Let $G$ be a finite group such that $\mathsf{Aut}(G)$ acts transitively on $G \setminus \{e\}$. Show that $G$ must be an abelian $p$-group, for some prime $p$.

*Extra credit problem.* Let $H$ be a Sylow $p$-subgroup of $S_n$. Describe explicitly the structure of $H$ as a semidirect product.