Math 6310, Assignment 1 Due in class: Wednesday, September 7

1. Let G be a group, and H a subgroup. Prove that the number of right cosets of H is equal to the number of left cosets.

2. Let G be an abelian group. Prove that $\{g \in G : |g| < \infty\}$ is a subgroup of G (called the *torsion subgroup* of G). Give an explicit example where this set is not a subgroup if G is non-abelian.

3. Let G be a group such that $x^2 = e$, for all $x \in G$. We saw in class that G must be abelian. Assuming that G is finite, prove that $|G|$ is a power of 2.

4. Let G be a group such that $(xy)^3 = x^3y^3$ for all $x, y \in G$, and the map $x \longmapsto x^3$ is bijective. Prove that G is abelian.

5. Let F be a field. Define the *Heisenberg group* over F:
$$H(F) = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in F \right\}.$$
   (a) Show that $H(F)$ is a group.
   (b) Is $H(F)$ abelian? What is the order of $H(F)$? What is the center $Z(H(F))$ of $H(F)$?
   (c) Find the order of every element in $H(\mathbb{Z}/2\mathbb{Z})$.
   (d) Show that every non-identity element in $H(\mathbb{R})$ has infinite order.

6. (a) In the group S_4, explicitly describe a subgroup of order 8.
   (b) In the group S_9, explicitly describe a subgroup of order 81.

   Hint: In S_4, it is possible to write a set of 8 permutations explicitly and show that the set is closed under multiplication, but in S_9 this approach looks impractical. Instead, try to construct these subgroups as centralizers, e.g., in S_4 one can consider the centralizer of the permutation $(12)(34)$, and argue that it has 8 elements (without writing them all down).

7. Prove that S_n is generated by:
   (a) transpositions $(12), (23), (34), \ldots, (n-1, n)$;
   (b) cycles (12) and $(123 \ldots n)$.

8. Determine, up to isomorphism, all groups of order less than or equal to 10.