6220. Problem Set 3

Due date: Wednesday, February 20.

Problem 1-4: (Computing integrals using residues) Rudin page 228: 8, 11, 13 and verify that

$$\int_0^\infty \frac{\sqrt{x}}{x^2 + 2x + 5} dx = \frac{\pi}{2\sqrt{2}} \sqrt{\sqrt{5} - 1}.$$

Problem 5: Rudin page 229 problem 22.

Problem 6: Locate the singularities of the following functions and classify them (removable, pole of order ?, essential):

(1) $\frac{z^2}{\sin z}$;
(2) $\cot \pi z$;
(3) $\frac{e^z - 1}{e^{2z} - 1}$.

Problem 7: (Singularities at ∞) Suppose that f is holomorphic for $|z| > R$ and set $g(z) = f(1/z)$. (So g is holomorphic for $|z| < 1/R$.) Then f is said to have a singularity at ∞ if g has a singularity at 0, and the singularity at ∞ is classified according to the classification of the singularity at 0.

(1) Show that f has a removable singularity at ∞ if and only if $|f(z)|$ is bounded for $|z| > R$.
(2) Show that f has a pole at ∞ of order m if and only if

$$h(z) = \frac{f(z)}{z^m}$$

has a removable singularity at ∞ and $\lim z \to \infty h(z)$ is nonzero.
(3) Show that if f has a nonessential singularity at ∞ and f is entire, then f must be a polynomial.
(4) Classify the nature of the singularity at ∞ for each of the following functions.
  (a) $\sin \frac{1}{z}$;
  (b) e^z;
  (c) $(z + \frac{1}{z})^3$.