6220. Problem Set 1

Due date: Tuesday, January 22.

Problem 1: (Holomorphicity vs. differential) If $f : \Omega \to \mathbb{C}$, $\Omega \subset \mathbb{C}$, then f can be regarded as $f : \Omega \to \mathbb{R}^2$, $\Omega \subset \mathbb{R}^2$, so it makes sense to talk about the differential of f and differentiability in the sense of Rudin 7.22.

(a) Prove that, if f is holomorphic, then it has a differential.
(b) Prove that, if f has a differential at every point in Ω, and the Cauchy-Riemann equations are satisfied at every point in Ω, then f is holomorphic in Ω.
(c) Give an example of a function $f : \Omega \to \mathbb{R}^2$, for which the Cauchy-Riemann equations are satisfied at $(x, y) = (0, 0)$, but f is not differentiable at $(0, 0)$.

Problem 2: (Rudin 2/227) Suppose that f is an entire function, and that in the power series around any point a,

$$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n,$$

at least one coefficient is 0. Prove that f is a polynomial.

Problem 3: Determine all values of i^i and $(-1)^i$.

Problem 4: Give an expression for $\arctan z$ in terms of the complex logarithm. (See Ahlfors, p. 47, for a similar discussion about $\arccos z$.)

Problem 5: Show that the roots of the binomial equation $z^n = a$, ($n \geq 3$ positive integer) are the vertices of a regular polygon (equal sides and angles).