Math 2210-1. Practice Test 2. Fall 2007.

Name: Solutions

October 18, 2007

Problem 1: _____ /40

Problem 2: _____ /40

Problem 3: _____ /40

Problem 4: _____ /30

Total: _____ /150

Instructions: The exam is closed book, closed notes and calculators are not allowed. You are only allowed one lettersize sheet of paper with anything on it.

You will have 50 minutes for this exam. The point value of each problem is written next to the problem - use your time wisely. Please show all work, unless instructed otherwise. Partial credit will be given only for work shown.

Problem 1(40 points) Consider the function $f(x,y)=x^2/2+y^2$.

(1) Find the equation the level curve for f that goes through the point $(4,1)$.
(2) Find the gradient vector ∇f at $(4,1)$.
(3) Draw the level curve and draw the gradient vector with its initial point at $(4,1)$.

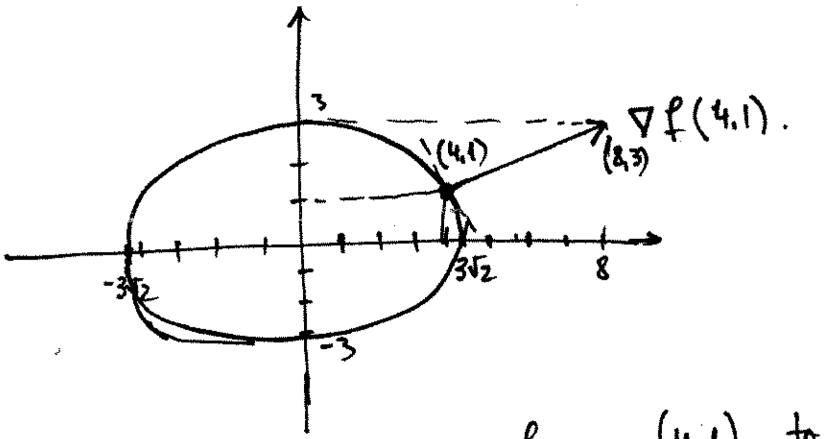
(1) $f(4,1) = 9$ so the level curve is $\boxed{\frac{x^2}{2}+y^2=9}$

(2) $\nabla f(4,1) = f_x(4,1)\,\vec{i} + f_y(4,1)\,\vec{j}$

$f_x = x \qquad f_y = 2y$

$\boxed{\nabla f(4,1) = 4\,\vec{i} + 2\,\vec{j}}$

(3) The level curve is an. ellipse. $\frac{x^2}{(3\sqrt{2})^2} + \frac{y^2}{3^2} = 1$.



The gradient $\nabla f(4,1)$ points away from $(4,1)$ to the point $(8,3)$ $((4,1)+(4,2)=(8,3))$. The gradient is perpendicular to the tangent line at $(4,1)$ to the ellipse.

Problem 2(40 points) Consider the function $f(x,y) = x^3 + y^3 - 6xy$.

(1) Find all critical points of $f(x,y)$.
(2) Decide for each critical point if it is a local maximum, local minimum, or a saddle point.
(3) Does f have a global maximum or a global minimum?

(1) The domain of the function is all x, y real numbers (unrestricted

Also $f(x,y)$ is differentiable everywhere. This means that the only critical points are those for which

$$\nabla f = 0.$$

$f_x = 3x^2 - 6y$

$f_y = 3y^2 - 6x$

We get the system $\begin{cases} 3x^2 - 6y = 0 & :3 \\ 3y^2 - 6x = 0 & :3 \end{cases}$ $\quad \begin{cases} x^2 - 2y = 0 \\ y^2 - 2x = 0. \end{cases}$

Solve for x in the second, substitute in the first.

$x = \dfrac{y^2}{2}$ and $\left(\dfrac{y^2}{2}\right)^2 = 2y \Leftrightarrow \begin{cases} y^4 = 8y \\ x = \dfrac{y^2}{2} \end{cases} \Rightarrow$ either $\begin{cases} y = 0 \\ x = 0 \end{cases}$ or $\begin{cases} y = 2 \\ x = 2 \end{cases}$.

Critical points are $\boxed{(0,0)}$, $\boxed{(2,2)}$.

(2) Apply the second partials test:

$f_{xx} = 6x$

$f_{xy} = -6$

$f_{yy} = 6y$

$$D(x,y) = \begin{vmatrix} \boxed{6x} & -6 \\ -6 & 6y \end{vmatrix} = 36(xy - 1)$$

$(0,0)$: $D(0,0) = -36 < 0 \Rightarrow (0,0)$ is a <u>saddle pt</u> (and not an extremum

$(2,2)$: $\left.\begin{array}{l} D(2,2) = 36 \cdot 3 > 0 \\ f_{xx}(2,2) = 12 > 0 \end{array}\right\} \Rightarrow$ $\boxed{(2,2) \text{ is a local minimum point}}$

(3) Every global extremum is a local extremum. Since there are no local maxima, there is $\boxed{\text{no global max.}}$ The only possibility would be for $(2,2)$ to give a global minimum: $f(2,2) = -6$

But we can see that $f(0,y) = y^3 \longrightarrow -\infty$ as $y \to -\infty$

so there $\boxed{\text{no global minimum}}$.

Problem 3(40 points)

(1) Find the limit

$$\lim_{(x,y)\to(0,0)} \frac{\tan(x^2+y^2)}{x^2+y^2}.$$

(2) Prove that the limit

$$\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y}$$

does not exist.

(1) Change to polar coordinates : $x = r\cos\theta$, $y = r\sin\theta$ $\quad x^2+y^2=r^2$

$(x,y)\to(0,0) \iff r\to 0$

The limit becomes:

$$\lim_{r\to 0} \frac{\tan r^2}{r^2} \underset{\substack{\uparrow \\ \frac{0}{0}\ \text{Apply l'Hôpital}}}{=} \lim_{r\to 0} \frac{\frac{1}{\cos r^2}\cdot \cancel{2r}}{\cancel{2r}} = \lim_{r\to 0} \frac{1}{\cos^2 r^2} = \frac{1}{\cos 0} = \boxed{1}.$$

(2) We show that the given limit depends on the path $(x,y)\to(0,0)$.

Approach $(0,0)$ on the line $y = mx$, $x\to 0$:

$$\lim_{\substack{y=mx \\ x\to 0}} \frac{x-y}{x+y} = \lim_{\substack{y=mx \\ x\to 0}} \frac{x-mx}{x+mx} = \lim_{x\to 0} \frac{(1-m)\cancel{x}}{(1+m)\cancel{x}}$$

$$= \frac{1-m}{1+m}$$

This clearly depends on m (e.g. if $m=0$ the limit is 1, if $m=1$ the limit is 0)

In conclusion

$$\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y} \ \boxed{\text{does not exist}}$$

Problem 4(30 points) Find the equation of the tangent plane to the surface $9x^2 + 4y^2 + 9z^2 = 34$ at the point $(1, 2, -1)$.

This is a particular case of the general problem of finding the tangent plane to the level surface

$$F(x,y,z) = k.$$

$$F(x,y,z) = 9x^2+4y^2+9z^2.$$

The gradient ∇F is normal to the tangent plane

$$\nabla F = 18x\,\vec{i} + 8y\,\vec{j} + 18z\,\vec{k}$$

$$\nabla F(1,2,-1) = 18\,\vec{i} + 16\,\vec{j} - 18\,\vec{k}.$$

Then the equation of the plane normal to $\nabla F(1,2,-1)$ and containing the point $(1,2,-1)$ is:

$$18(x-1) + 16(y-2) - 18(z+1) = 0$$

$$\boxed{18x + 16y - 18z = 68}$$