Math 2210-1. Practice Test 1. Fall 2007.

Name: Solutions September 15, 2007

Problem 1: ______ /40

Problem 2: ______ /20

Problem 3: ______ /30

Problem 4: ______ /30

Problem 5: ______ /30

Total: ______ /150

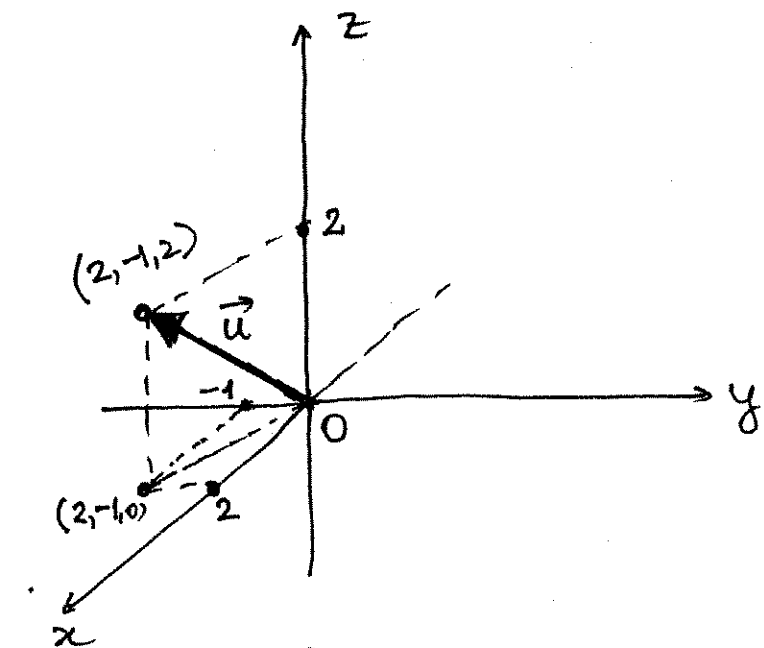
Instructions: The exam is closed book, closed notes and calculators are not allowed. You are only allowed one A4-size sheet of paper with anything on it.

You will have 50 minutes for this exam. The point value of each problem is written next to the problem - use your time wisely. Please show all work, unless instructed otherwise. Partial credit will be given only for work shown.

Problem 1(40 points) Let $\vec{u} = 2\vec{i} - \vec{j} + 2\vec{k}$ and $\vec{v} = 5\vec{i} + \vec{j} - 3\vec{k}$ be two vectors.

(1) Draw a sketch of vector $\vec{u}$ in the xyz-coordinate system.
(2) Find the cosine of the angle between $\vec{u}$ and $\vec{v}$.
(3) Find the area of the triangle which has two sides $\vec{u}$ and $\vec{v}$.

(1)



(2) $$\cos(\text{angle}) = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|} = \frac{10-1-6}{\sqrt{9}\,\sqrt{35}} = \frac{1}{\sqrt{35}}$$

(3) $$\text{area}(\text{triangle}) = \frac{1}{2}\|\vec{u}\times\vec{v}\|$$

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 2 \\ 5 & 1 & -3 \end{vmatrix} = \vec{i} + 16\vec{j} + 7\vec{k}$$

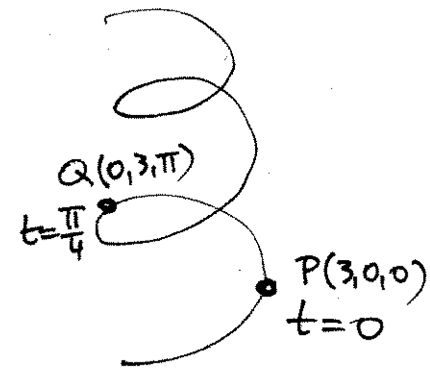
$$\text{area} = \frac{1}{2}\sqrt{1+256+49} = \frac{1}{2}\sqrt{306}$$

Problem 2(20 points) Find the length of the curve

$$\vec{r}(t) = 3\cos(2t)\vec{i} + 3\sin(2t)\vec{j} + 4t\vec{k},$$

between the points $P(3,0,0)$ and $Q(0,3,\pi)$.

We need to find the values of t for which points P and Q are obtained.



$P(3,0,0)$: $\quad 3\cdot\cos 2t = 3$, $\quad 3\sin 2t = 0$, $\quad 4t = 0 \implies t = 0$

$Q(0,3,\pi)$: $\quad 3\cos 2t = 0$, $\quad 3\sin 2t = 3$, $\quad 4t = \pi \implies t = \frac{\pi}{4}$.

So $\text{length(curve)} = \int_0^{\frac{\pi}{4}} \|\vec{v}(t)\|\, dt.$

$$\|\vec{v}(t)\| = \| -6\sin(2t)\vec{i} + 6\cos(2t)\vec{j} + 4\vec{k}\| = \sqrt{36+16} = \sqrt{52}$$

$$\text{length (curve)} = \int_0^{\frac{\pi}{4}} \sqrt{52}\, dt = \frac{\pi}{4}\sqrt{52}$$

Problem 3(30 points) Consider the position vector

$$\vec{r}(t) = (\cos t + \sin t)\vec{i} + (\sin t - \cos t)\vec{j}.$$

(1) Find the acceleration $\vec{a}(t)$.
(2) Find the normal and tangential components of the acceleration a_N and a_T.
(3) What curve does $\vec{r}(t)$ describe?

(1) $\vec{a}(t) = \vec{r}''(t)$ $\quad \vec{r}'(t) = (-\sin t + \cos t)\vec{i} + (\cos t + \sin t)\vec{j}$

$\vec{a}(t) = (-\cos t - \sin t)\vec{i} + (-\sin t + \cos t)\vec{j}$

(2) First method: Note from (1) that $\vec{a}(t) = -\vec{r}(t)$ and $\vec{r}'(t)$ is perpendicular to $\vec{a}(t)$.

This means that there is only a normal component: $(\vec{a} = a_N \vec{N})$ $\quad a_T = 0, \; a_N = \|\vec{a}\|$, $\; a_N = \sqrt{2}$

Second method: find $\vec{T}, \vec{N}$ etc.

$\vec{T} = \dfrac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$ $\quad \|\vec{r}'(t)\| = \sqrt{(-\sin t + \cos t)^2 + (\cos t + \sin t)^2} = \sqrt{2}$

$\vec{T} = \frac{1}{\sqrt{2}} (-\sin t + \cos t)\vec{i} + (\cos t + \sin t)\vec{j}$

$a_T = \vec{T} \cdot \vec{a} = 0.$

$\vec{T}' = \frac{1}{\sqrt{2}} (-\cos t - \sin t)\vec{i} + (-\sin t + \cos t)\vec{j}$

$\vec{N} = \dfrac{\vec{T}'}{\|\vec{T}'\|} = \vec{T}'$ (since $\vec{T}'$ has length 1 in this case)

$a_N = \vec{N} \cdot \vec{a} = \frac{1}{\sqrt{2}} \Big[\underbrace{(\cos t + \sin t)^2 + (-\sin t + \cos t)^2}_{2}\Big] = \sqrt{2}$

(3) The curve is a circle (we can see it from the results about acceleration or $x^2 + y^2 = (\cos t + \sin t)^2 + (\sin t - \cos t)^2 = 2$

Problem 4(30 points)

(1) Find the parametric equations of the line containing the point $P(0,1,2)$ and parallel to the planes $2x - y + 3z = 5$ and $-x + 2y + 2z = 3$.

(2) Is this line parallel also to the plane $x + 3y - z = 2$?

(1) If the line is parallel to the two planes, then it must be perpendicular on both normal vectors:

$\vec{n}_1 = 2\vec{i} - \vec{j} + 3\vec{k}$ $\quad \vec{n}_2 = -\vec{i} + 2\vec{j} + 2\vec{k}$

So it is parallel to $\vec{n}_1 \times \vec{n}_2$:

$$\underbrace{\vec{n}_1 \times \vec{n}_2}_{\vec{v}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 3 \\ -1 & 2 & 2 \end{vmatrix} = (-8)\vec{i} - 7\vec{j} + 2\vec{k}$$

The line determined by the point $P(0,1,2)$ and $\vec{v} = -8\vec{i} - 7\vec{j} + 2\vec{k}$ has parametric equations:

$$\begin{cases} x = 0 - 8t \\ y = 1 - 7t \\ z = 2 + 2t \end{cases} \quad t \text{ parameter}$$

(2) The line would be parallel to the plane $x + 3y - z = 2$ if and only if $\vec{v}$ is perpendicular to $\vec{n} = \vec{i} + 3\vec{j} - \vec{k}$

$$\vec{v} \cdot \vec{n} = (-8\vec{i} - 7\vec{j} + 2\vec{k}) \cdot (\vec{i} + 3\vec{j} - \vec{k})$$

$$= -8 - 21 - 2 \neq 0 \quad \text{so } \underline{\text{no}}$$

Problem 5(30 points) Consider the surface given by the equation

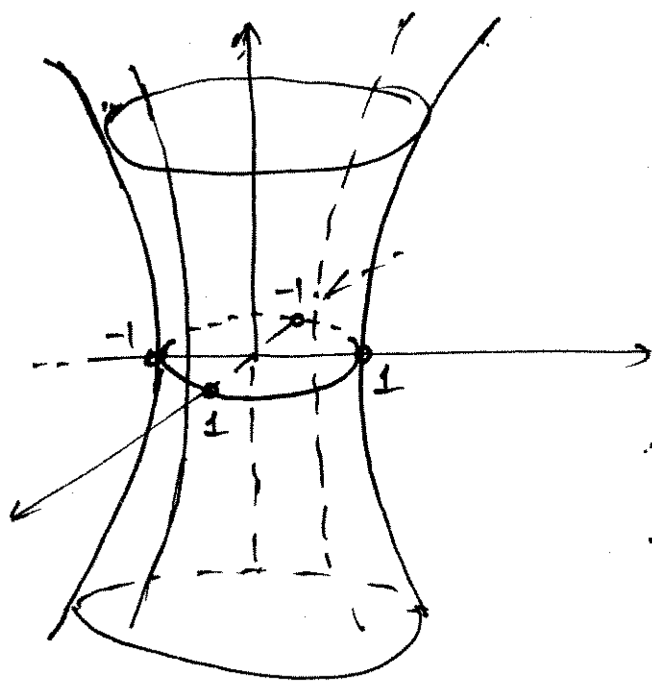
$$x^2 + y^2 - 3z^2 = 1.$$

(1) Graph the surface by hand showing the intersections (traces) with the coordinate planes.
(2) Write the equation of the surface in spherical coordinates.

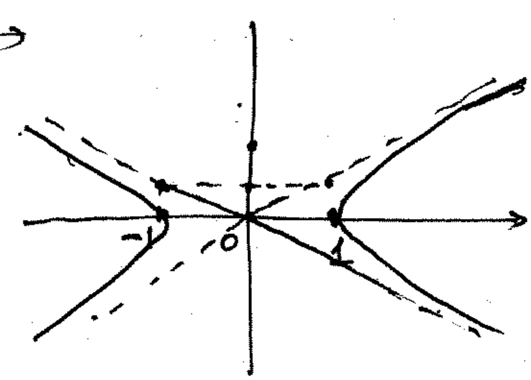
(1) (We can recognize that this is a hyperboloid of one sheet):

traces:

$x=0$	$y^2 - 3z^2 = 1$	hyperbola
$y=0$	$x^2 - 3z^2 = 1$	hyperbola
$z=0$	$x^2 + y^2 = 1$	circle



actually it should be flatter b/c: $y^2 - 3z^2 = 1$ is more like:



(2). In spherical coordinates: $x^2+y^2 = \rho^2 \sin^2\phi$
$z^2 = \rho^2 \cos^2\phi$

$$\rho^2 (\sin^2\phi - 3\cos^2\phi) = 1$$