Math 2210-1. Practice Final. Fall 2007.

Name: Solutions   December 6, 2007

Problem 1: _____ /70

Problem 2: _____ /65

Problem 3: _____ /65

Problem 4: _____ /60

Problem 5: _____ /40

Total: _____ /300

Instructions: The exam is closed book, closed notes and calculators are not allowed. You are only allowed four letter-size sheets of paper with anything on it.

You will have two hours for this test. The point value of each problem is written next to the problem - use your time wisely. Please show all work, unless instructed otherwise. Partial credit will be given only for work shown.

Problem 1.

(1) Find the equation of the tangent plane to the surface $x^2+y^2-z^2=4$ at the point $(2,1,1)$.
(2) Find the parametric equations for the tangent line to the curve $\vec{r}(t)=t\vec{i}+\frac{1}{2}t^2\vec{j}+\frac{1}{3}t^3\vec{k}$, at the point $P(2)=(2,2,\frac{8}{3})$.
(3) Is the line in (2) parallel to the plane in (1)?

(1) Equation of tangent plane to $F(x,y,z)=\overset{\text{constant}}{C}$ at (x_0,y_0,z_0):

$$F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)=0$$

In our case $F(x,y,z)=x^2+y^2-z^2$ and the point is $(2,1,1)$

$F_x=2x \quad F_y=2y \quad F_z=-2z$. So the plane is:

$$4(x-2)+2(y-1)-2(z-1)=0 \qquad 4x+2y-2z=8$$

$$\boxed{2x+y-z=4}$$

(2) ~~The~~ vector in the direction of the tangent line is the tangent vector ~~of~~ $\vec{r}'(t)$ at $t=2$
(velocity) $\vec{r}'(t)=\vec{i}+t\vec{j}+t^2\vec{k} \qquad \vec{r}'(2)=\vec{i}+2\vec{j}+4\vec{k}$

Parametric equations of the line:

$$\begin{cases} x=2+t \\ y=2+2t \\ z=\frac{8}{3}+4t \end{cases}$$

(3) ~~The~~ normal vector to the plane in (1) is
$\vec{n}=2\vec{i}+\vec{j}-\vec{k}$
The vector in the direction of the line from (2) is
$\vec{v}=\vec{r}'(2)=\vec{i}+2\vec{j}+4\vec{k}$
Since $\vec{n}\cdot\vec{v}=2+2-4=0$, it follows that $\vec{v}$ is parallel to the plane, so the line is parallel to the plane

(3) From (1), (2) it follows that the global max of $f(x,y)$ is $\boxed{\frac{27}{2}}$ at $\left(\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right)$ and $\left(-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$
and the global min is $\boxed{0}$ at $(0,0)$

Problem 2. Consider the function $f(x, y) = x^2 + y^2 - xy$ on the closed disk $S : x^2 + y^2 \leq 9$.

(1) Find the critical points in the interior of S and decide if they give local maxima, minima, or neither.

(2) Find the maximum and minimum values of $f(x, y)$ on the boundary of S.

(3) Using the results in (1) and (2), conclude what the global maximum and minimum values of $f(x, y)$ on S are.

(1) The interior of S is $x^2+y^2<9$.

Critical points: $\nabla f = 0$ $f_x = 2x-y$, $f_y = 2y-x$ so we get the system $\begin{cases} 2x-y=0 \\ 2y-x=0 \end{cases}$

$\Rightarrow \begin{cases} 2x=y \\ x=2y \end{cases} \Rightarrow 4y=y$ $y=0$, $x=0$ $\boxed{(0,0)}$ only critical point in int S.

We apply the second partials test:

$f_{xx}=2$ $f_{xy}=f_{yx}=-1$ $f_{yy}=2$ $D(x,y) = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4-1 = 3>0$

$f_{xx}(0,0)=2>0$ $D(0,0)=3>0$

$\Rightarrow$ $(0,0)$ gives local minimum, $f(0,0)=0$

(2) We use Lagrange multipliers for $f(x,y)=x^2+y^2-xy$ subject to the constraint $x^2+y^2=9$

$g(x,y) = x^2+y^2-9$

$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y)=0 \end{cases} \Leftrightarrow \begin{cases} 2x-y = \lambda\cdot 2x \\ 2y-x=\lambda\cdot 2y \\ x^2+y^2=9 \end{cases}$ substract $3(x-y) = 2\lambda(x-y)$

two cases: $x=y$ or $\lambda = \frac{3}{2}$

• $x=y$ then $\begin{cases} x=\lambda\cdot 2x \\ 2x^2=9 \end{cases}$ so $\lambda=\frac{1}{2}$, $x=\pm\frac{3}{\sqrt{2}}$ $\left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right), \left(-\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right)$

• $\lambda=\frac{3}{2}$ then $\begin{cases} 2x-y=3x \\ x^2+y^2=9 \end{cases}$ so $\begin{cases} y=-x \\ 2x^2=9 \end{cases}$ $\left(\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right), \left(-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$

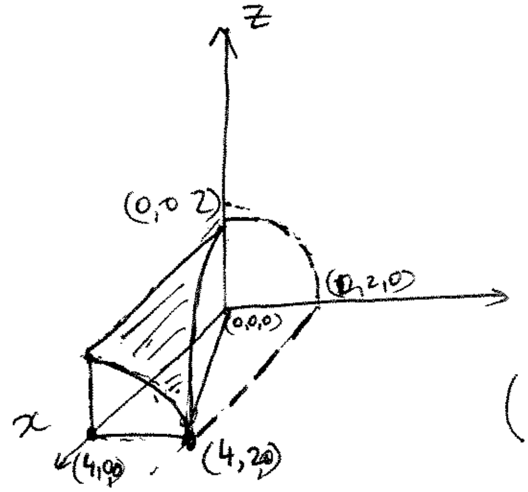
The values of $f(x,y)$ are:

$\left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$	$\left(-\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right)$	$\left(\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right)$	$\left(-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$
min $\rightarrow \frac{9}{2}$	min $\frac{9}{2}$	$\frac{27}{2}$ max	$\frac{27}{2} \leftarrow$ max

Problem 3.

(1) Draw a clear picture of the wedge cut from the cylinder $y^2 + z^2 = 4$ in the first octant by the planes $y = \frac{x}{2}$ and $x = 4$.
(2) Find the volume of the cylindrical wedge described in (1). Evaluate the integral!
(3) Set up completely an integral which calculates the cylindrical surface area of the wedge in (1). Do not evaluate the integral! (*Extra credit*: Evaluate the integral.)

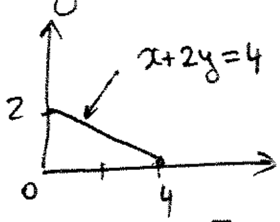
(1)



(We'll use cartesian coordinates; try cylindrical triple integral for an (easier) alternative)

(2) $z=\sqrt{4-y^2}$ Volume $V=\iint_S z\,dA = \iint_S \sqrt{4-y^2}\,dA$

S triangle region in xy-plane:



$$V=\int_0^2\int_0^{4-2y}\sqrt{4-y^2}\,dx\,dy = \int_0^2 (4-2y)\sqrt{4-y^2}\,dy$$

$y = 2\sin\theta$

$dy = 2\cos\theta\,d\theta$

$$V=2\int_0^{\frac{\pi}{2}}(4-4\sin\theta)\cos^2\theta\,d\theta = 8\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta - 8\int_0^{\frac{\pi}{2}}\sin\theta\cos^2\theta\,d\theta$$

$$V = \underbrace{4\int_0^{\frac{\pi}{2}}(1+\cos 2\theta)\,d\theta}_{4\cdot\frac{\pi}{2} + \cancel{2\cdot\sin 2\theta\Big]_0^{\frac{\pi}{2}}}} + 8\left[\frac{1}{3}\cos^3\theta\right]_0^{\frac{\pi}{2}} = \boxed{V=2\pi-\frac{8}{3}}$$

(3) $f(x,y)=\sqrt{4-y^2}$ $f_x=0$ $f_y=\frac{-y}{\sqrt{4-y^2}}$ so $\sqrt{f_x^2+f_y^2+1} = \frac{2}{\sqrt{4-y^2}}$

Then: cyl Area $= \iint_S \frac{2}{\sqrt{4-y^2}}\,dA = \boxed{\int_0^2\int_0^{4-2y}\frac{2}{\sqrt{4-y^2}}\,dx\,dy}$

Problem 4. Let C be a curve between the points $(0,0)$ and $(1,2)$ and consider the line integral

$$\int_C y^2 dx + 2xy dy.$$

(1) Show that the line integral is independent of the path, i.e., independent of the particular form of C.
(2) Evaluate the integral by choosing a particular (simple) C.
(3) Evaluate the integral by using the fundamental theorem of calculus for line integrals.

(1) This is true if and only if the vector field is <u>conservative</u>.
Since the domain of $\vec{F}$ is the whole plane, $\vec{F}(x,y) = y^2\vec{\imath} + 2xy\vec{\jmath}$
this is equivalent to $\vec{F}$ being irrotational.
Let's compute the curl : $\operatorname{curl}\vec{F} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\vec{k}$

$\operatorname{curl}\vec{F} = (2y - 2y)\vec{k} = \vec{0}$

so indeed $\vec{F}$ is irrotational (therefore conservative)

(2) We can choose any C, so we'll choose the straight line between $(0,0)$ and $(1,2)$. The parametric equations are $C : \begin{cases} x = t \\ y = 2t \end{cases}$ $0 \le t \le 1$

$$\int_C y^2 dx + 2xy\, dy = \int_0^1 4t^2 dt + 4t^2 \cdot 2dt$$

$$= \int_0^1 12t^2 dt = 4t^3\Big]_0^1 = \boxed{4}$$ ✓

(3) To do this, we need to find the potential $f(x,y)$ whose gradient is $\vec{F}(x,y)$: $\nabla f = \vec{F}$

$f_x = y^2 \Rightarrow f(x,y) = xy^2 + c_1(y)$ so $f(x,y) = xy^2$ works.

$f_y = 2xy \Rightarrow f(x,y) = xy^2 + c_2(x)$

Then $\int_C y^2 dx + 2xy\, dy = f(1,2) - f(0,0) = 4 - 0$

$= \boxed{4}$ ✓

Problem 5. Let $\vec{F}$ be the vector field

$$\vec{F} = xy\vec{i} + (x^2+y^2)\vec{j}.$$

(1) Compute $curl\vec{F}$ and $div\vec{F}$.
(2) Calculate the flux of $\vec{F}$ across the triangle with vertices $(0,0)$, $(1,1)$, and $(2,0)$.

(1) $\text{curl}\,\vec{F} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\vec{k} = (2x - x)\vec{k}$

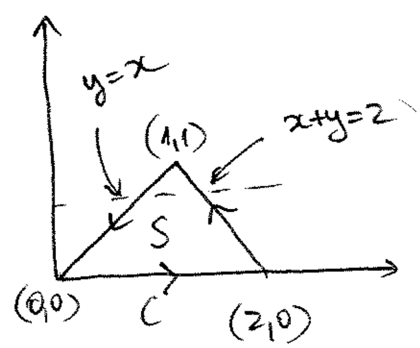
$\boxed{\text{curl}\,\vec{F} = x\vec{k}}$

$M(x,y) = xy$

$N(x^2+y^2) = x^2+y^2$

$\text{div}\,\vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = y + 2y \Rightarrow \boxed{\text{div}\,\vec{F} = 3y}$

(2)



The flux is $\int_C \vec{F}\cdot\vec{n}\,ds$ where $\vec{n}$ is the normal vector at (x,y).

(Note that this can be computed directly by breaking it into a sum of three line integrals, one for each edge of the triangle).

We use Green's theorem (actually in the form of Gauss's diverg

$$\int_C \vec{F}\cdot\vec{n}\,ds = \iint_S \text{div}\,\vec{F}\,dA = \iint_S 3y\,dA$$

$$= \int_0^1 \int_y^{2-y} 3y\,dx\,dy = 3\int_0^1 y(2-y-y)\,dy$$

$$= 6\int_0^1 (y-y^2)\,dy$$

$$= 6\left[\frac{1}{2} - \frac{1}{3}\right] = \boxed{1}$$