Here are the solutions to the more interesting questions (for which solutions were not presented in class).

(2) Prove that if \( n \) is a perfect square, then \( n + 2 \) is not a perfect square.

**Proof.** We will prove it by contradiction. Assume that both \( n \) and \( n + 2 \) are perfect squares. This means there exist nonnegative integers \( a, b \) such that \( n = a^2 \) and \( n + 2 = b^2 \). Then \( 2 = (n + 2) - n = b^2 - a^2 = (b-a)(b+a) \). Therefore \( b + a = 2 \) and \( b - a = 1 \), so \( 2b = 3 \), and \( b = 3/2 \), contradiction with the fact that \( n \) is an integer. \( \square \)

(10) Show that among any group of 10 (not necessarily consecutive) integers, there are two which are congruent mod 8.

**Proof.** Consider the residues mod 8: 0, 1, 2, \ldots, 7. There are 8 of them, and 10 numbers. By the pigeonhole principle, two of the numbers must have the same residue mod 8, which is equivalent to saying that they are congruent mod 8. \( \square \)

(11) Show that \( \binom{2n}{2} = 2 \binom{n}{2} + n^2 \).

**Proof.** One may verify this algebraically by using that \( \binom{k}{2} = \frac{k(k-1)}{2} \), for \( k = 2n \) and \( k = n \).

A different solution is by counting. Assume you have \( 2n \) balls, and \( n \) are red and \( n \) are blue. There are \( \binom{2n}{2} \) ways to choose 2 balls out of these \( 2n \). This is the left hand side of the identity. But we count this differently by considering colors: there are \( \binom{n}{2} \) ways to choose two red balls, \( \binom{n}{2} \) ways to choose two blue balls, and \( n \cdot n \) ways to choose one red and one blue. This accounts for the right hand side of the identity. \( \square \)

(12) Let \( N \geq 2 \) be a positive number, and assume a family has \( N \) children. Consider the events \( E \), the family has children of both sexes, and \( F \), the family has at most one boy. Prove that
(1) $E$ and $F$ are independent when $N = 3$;
(2) $E$ and $F$ are not independent when $N = 4$.

**Proof.** (a) $p(E) = 6/8$, $p(F) = 4/8$, $p(E \cap F) = 3/8$, and notice that $p(E \cap F) = p(E) \cdot p(F)$.
   (b) $p(E) = 14/16$, $p(F) = 5/16$, $p(E \cap F) = 4/16$, notice that $p(E)p(F) = 5/64 \neq p(E \cap F) = 14/16$. \qed

(13) In a five-card poker hand, prove that it is more likely to have a full house than four of a kind.

**Proof.** Let $E$ be the event that one gets a full house, and $F$ the event of four of a kind. The probabilities are:

$$p(E) = \frac{P(13, 2)C(4, 3)C(4, 2)}{C(52, 5)} = \frac{13 \cdot 12 \cdot 4 \cdot 6}{C(52, 5)} = \frac{6 \cdot 13 \cdot 48}{C(52, 5)}$$

$$p(F) = \frac{C(13, 1)C(4, 4)C(48, 1)}{C(52, 5)} = \frac{13 \cdot 48}{C(52, 5)},$$

so we see that it is six times more likely to have a full house than four of a kind. \qed