Math 2200. Discrete Mathematics.
Topics for the final

The final exam is scheduled for Wednesday, 12/17, 10:30-12:30 pm, in the usual classroom. You are allowed three pages with anything on them, no books or calculators. There will be six problems in the final exam. We will ask you to give careful short proofs for various statements related to the main topics covered in this course. You may use direct proofs, contraposition, proofs by contradiction, examples/counterexamples, truth or membership tables, counting to prove these statements.

Here are some representative examples:

(1) Prove that the first Morgan law $\overline{p \wedge q} \equiv \overline{p} \vee \overline{q}$ holds.

(2) Prove that if n is a perfect square, then $n+2$ is not a perfect square.

(3) Prove that

$$1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

for every positive integer n.

(4) If n is a positive integer not prime, then n has a prime divisor less than or equal to $\sqrt{n}$.

(5) If $a = bq + r$, where a, b, q, r are integers, then $\gcd(a, b) = \gcd(b, r)$.

(6) Show that the system of congruences

$$\begin{cases} x \equiv 2 (\text{mod } 6) \\ x \equiv 3 (\text{mod } 9) \end{cases}$$

has no solutions.

(7) Show that the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = ax + b$ is invertible, where a and b are constants, with $a \neq 0$.

(8) Show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ for any three sets A, B, C.

(9) Prove that is $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ are sets such that $A_j \subseteq B_j$ for all $j = 1, 2, \ldots, n$, then

$$\bigcup_{j=1}^{n} A_j \subseteq \bigcup_{j=1}^{n} B_j.$$

(10) Show that among any group of 10 (not necessarily consecutive) integers, there are two which are congruent mod 8.

(11) Show that $\binom{2n}{2} = 2\binom{n}{2} + n^2$.

(12) Let $N \geq 2$ be a positive number, and assume a family has N children. Consider the events E, the family has children of both sexes, and F, the family has at most one boy. Prove that
   (a) E and F are independent when $N = 3$;
   (b) E and F are not independent when $N = 4$.

(13) In a five-card poker hand, prove that it is more likely to have a full house than four of a kind.