Math 2200-1. Practice Final. Solutions. Fall 2008.

December 14, 2008

Problem 1. Prove that

(a) $((p \vee q) \wedge (\overline{p} \vee r)) \to (q \vee r)$ is a tautology.

(b) $(\overline{p} \wedge (p \to q)) \to \overline{q}$ is not a tautology.

Proof. (a) We may use a truth table.

p	q	r	$p \vee q$	$\overline{p}$	$\overline{p} \vee r$	$(p \vee q) \wedge (\overline{p} \vee r)$	$q \vee r$	full
T	T	T	T	F	T	T	T	T
T	T	F	T	F	F	F	T	T
T	F	T	T	F	T	T	T	T
T	F	F	T	F	F	F	F	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T	T
F	F	T	F	T	T	F	T	T
F	F	F	F	T	T	F	F	T

(b) If p is false and q is true, then $\overline{p}$ is true and $p \to q$ is true, so $\overline{p} \wedge (p \to q)$ is true. But $\overline{q}$ is false, which makes the full compound proposition false. (This example is one of the most common mistakes in the application of the rules of inference.)

□

Problem 2. Prove by mathematical induction that

$$\frac{1}{2n} \leq \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot (2n)}, \quad \text{for all } n \geq 1.$$

Proof. The base case is $n = 1$, which gives $\frac{1}{2} \leq \frac{1}{2}$, true.

Assume the inequality holds for $n = k$, and let us prove it for $n = k + 1$. The induction hypothesis says then

$$\frac{1}{2k} \leq \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2k-1)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot (2k)},$$

and we multiply it on both sides by $\frac{2(k+1)-1}{2(k+1)} = \frac{2k+1}{2k+2}$. We see that it is sufficient to prove that $\frac{1}{2k+2} \leq \frac{1}{2k} \cdot \frac{2k+1}{2k+2}$. If we multiply by $2k(2k+2)$, this is equivalent with $2k \leq 2k+1$, which is true.

□

Problem 3.

(a) Prove that $(A_1 \cap A_2) \cup B = (A_1 \cup B) \cap (A_2 \cup B)$, for every sets A_1, A_2, B.

(b) Prove by mathematical induction that $(A_1 \cap A_2 \cap \dots \cap A_n) \cup B = (A_1 \cup B) \cap (A_2 \cup B) \cap \dots \cap (A_n \cup B)$, for every sets $A_1, A_2, \dots, A_n$, $n \geq 2$, and B.

Proof. (a) We can prove this by using a membership table, or by "double inclusion".

(b) For $n = 2$, this is part (a). Assume the identity holds for k sets, and let us prove it for $k+1$ sets:

$$\begin{aligned}(A_1 \cap A_2 \cap \cdots \cap A_{k+1}) \cup B &= ((A_1 \cap A_2 \cap \cdots \cap A_k) \cap A_{k+1}) \cup B \\ &= \text{(by part (a))}\ ((A_1 \cap A_2 \cap \cdots \cap A_k) \cup B) \cap (A_{k+1} \cup B) \\ &= \text{(by induction hypothesis)}\ (A_1 \cup B) \cap (A_2 \cup B) \cap \cdots \cap (A_k \cup B) \cap (A_{k+1} \cup B).\end{aligned}$$

□

Problem 4.

(a) Prove that if n is a positive integer, then n is even if and only if $7n + 4$ is even.

(b) Let x be a real number. Prove that is x^3 is irrational, then x is irrational. Is the converse true?

Proof. (a) We prove first the direct implication. Assume n is even. Then $n = 2k$ for some integer k. Then $7n + 4 = 14k + 4 = 2(7k + 2)$, which is even. For the converse, which is "if $7n + 4$ is even, then n is even", we use a proof by contrapositive. The contrapositive is: "if n is not even (that is, odd), then $7n + 4$ is not even (that is, odd)". If n is odd, then $n = 2k + 1$, for some integer k. Then $7n + 4 = 14k + 11 = 2(7k + 5) + 1$, which is odd.

(b) We use a proof by contrapositive: if x is rational, then x^3 is rational. If x is rational, then we may write $x = \frac{m}{n}$, where m, n are integers, and $n \neq 0$. Then $x^3 = \frac{m^3}{n^3}$, and $n^3 \neq 0$ (since $n \neq 0$). Clearly, m^3, n^3 are also integers; this proves x^3 is rational.

The converse is: if x is irrational, then x^3 is irrational. This is easily seen to be false, a counterexample is $x = \sqrt[3]{2}$ which is irrational, but $x^3 = 2$ is rational. □

Problem 5. Show that the system of congruences

$$\begin{cases} x \equiv 2 \ (\text{mod}3) \\ x \equiv 1 \ (\text{mod}4) \\ x \equiv 3 \ (\text{mod}5) \end{cases}$$

admits solutions by finding one explicit solution.

Proof. Since the moduli are all prime to each other, the chinese remainder theorem guarantees the existence of a solution, and moreover, gives an algorithm to find it.

m	3	4	5
M	20	15	12
M^{-1} mod m	2	3	3
a	2	1	3

A solution is $x = 2 \cdot 2 \cdot 20 + 1 \cdot 3 \cdot 15 + 3 \cdot 3 \cdot 12 = 80 + 45 + 108 = 233$. The smallest nonnegative solution (which is unique mod $3 \cdot 4 \cdot 5 = 60$) is $x = 53$.

□

Problem 6. Assume that we choose a random permutation of the five digits 12345. Show, by computing the probabilities involved, that it is less likely that this permutation contains at least one of the strings 12 and 43 than the permutation does not contain any of the strings 321 or 45.

Proof. Define the following events: E=the permutation contains at least one of the strings 12 and 43, F=the permutation does not contain any of the strings 321 or 45. We want to prove that

$$p(E) < p(F).$$

We compute $p(E)$ and $p(F)$. By inclusion-exclusion, $p(E) = \frac{4!+4!-3!}{5!} = \frac{42}{120}$.

For the other one, $p(F) = 1 - p(\overline{F}) = 1 - \frac{3!+4!-2!}{5!} = 1 - \frac{28}{120} = \frac{92}{120}$.

□