
Name: ____________________________  December 11, 2008

Problem 1: _____
Problem 2: _____
Problem 3: _____
Problem 4: _____
Problem 5: _____
Problem 6: _____

Total: _____

Instructions: The exam is closed book, closed notes and calculators are not allowed. You are only allowed three letter-size sheets of paper with anything on them.

You will have 120 minutes for this test. The point value of each problem is written next to the problem - use your time wisely. Please show all work, unless instructed otherwise. Partial credit will be given only for work shown.
Problem 1. Prove that
   (a) \((p \lor q) \land (\neg p \lor r)\) \(\rightarrow\) \((q \lor r)\) is a tautology.
   (b) \((\neg p \land (p \rightarrow q))\) \(\rightarrow\) \(\neg q\) is not a tautology.

Problem 2. Prove by mathematical induction that
   \[
   \frac{1}{2n} \leq \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2 \cdot 4 \cdot 6 \cdots (2n)}, \quad \text{for all } n \geq 1.
   \]

Problem 3.
   (a) Prove that \((A_1 \cap A_2) \cup B = (A_1 \cup B) \cap (A_2 \cup B)\), for every sets \(A_1, A_2, B\).
   (b) Prove by mathematical induction that \((A_1 \cap A_2 \cap \cdots \cap A_n) \cup B = (A_1 \cup B) \cap (A_2 \cup B) \cap \cdots \cap (A_n \cup B)\), for every sets \(A_1, A_2, \ldots, A_n, n \geq 2,\) and \(B\).

Problem 4.
   (a) Prove that if \(n\) is a positive integer, then \(n\) is even if and only if \(7n + 4\) is even.
   (b) Let \(x\) be a real number. Prove that if \(x^3\) is irrational, then \(x\) is irrational. Is the converse true?

Problem 5. Show that the system of congruences
   \[
   \begin{align*}
   x \equiv 2 \pmod{3} \\
   x \equiv 1 \pmod{4} \\
   x \equiv 3 \pmod{5}
   \end{align*}
   \]
   admits solutions by finding one explicit solution.

Problem 6. Assume that we choose a random permutation of the five digits 12345. Show, by computing the probabilities involved, that it is less likely that this permutation contains at least one of the strings 12 and 43 than the permutation does not contain any of the strings 321 or 45.