Math 2200-1. Solutions for Practice Quiz 2. Fall 2008.

Problem 1. The sequence of *Lucas numbers* is defined recursively by $\ell_0 = 2$, $\ell_1 = 1$, and $\ell_n = \ell_{n-1} + \ell_{n-2}$, for $n \geq 2$.

(1) Find ℓ_8.

(2) Show by induction that

$$\ell_0^2 + \ell_1^2 + \cdots + \ell_n^2 = \ell_n \ell_{n+1} + 2,$$

whenever n is a nonnegative integer.

Proof. (1) We compute recursively: $\ell_0 = 2$, $\ell_1 = 1$, $\ell_3 = 3$, $\ell_4 = 4$, $\ell_5 = 7$, $\ell_6 = 11$, $\ell_7 = 18$, $\boldsymbol{\ell_8 = 29}$.

(2) The base case is $n = 0$, which gives $2^2 = 2 + 2$. Assume the induction hypothesis $\ell_0^2 + \ell_1^2 + \cdots + \ell_k^2 = \ell_k \ell_{k+1} + 2$, and we would like to prove that $\ell_0^2 + \ell_1^2 + \cdots + \ell_{k+1}^2 = \ell_{k+1}\ell_{k+2} + 2$. Add ℓ_{k+1}^2 on both sides of the induction hypothesis, and we find that

$$\begin{aligned}
\ell_0^2 + \ell_1^2 + \cdots + \ell_k^2 + \ell_{k+1}^2 &= \ell_k \ell_{k+1} + 2 + \ell_{k+1}^2 \\
&= \ell_{k+1}(\ell_k + \ell_{k+1}) + 2 \\
&= \ell_{k+1}\ell_{k+2} + 2 \quad \text{(using the recursion formula)}
\end{aligned}$$

$\square$

Problem 2. Prove by induction that 6 divides $n^3 - n$ whenever n is a nonnegative integer.

Proof. The base case is $n = 0$. Then $n^3 - n = 0$, and 6 divides it. Let's assume 6 divides $k^3 - k$ and prove that 6 divides $(k+1)^3 - (k+1)$. By using the binomial formula $(k+1)^3 = k^3 + 3k^2 + 3k + 1$, we find that

$$(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k - k = (k^3 - k) + 3k(k+1).$$

$\square$

The term $(k^3 - k)$ is divisible by 6 by the induction hypothesis. The term $3k(k+1)$ is clearly divisible by 6, but since it also contains a product of two consecutive numbers, it must be divisible by 2,a s well, so by 6. Therefore the LHS is also divisible by 6, being a sum of two multiples of 6.

Problem 3. For which nonnegative integers n is $n! \geq n^2$? Prove your answer by induction.

Proof. We try to verify directly the inequality for $n = 0, 1, 2, 3, 4, 5$. It holds for $n = 0$ and $n = 1$, and then it seems to hold again for $n \geq 4$. Let's prove by induction that this holds for $n \geq 4$. The base case $n = 4$ gives $24 \geq 16$, which is true. Assume $k! \geq k^2$ for some $k \geq 4$, and let's deduce that $(k+1)! \geq (k+1)^2$. Multiply both sides of the induction hypothesis by $(k+1)$, and find that $(k+1)! \geq k^2(k+1)$. If we prove that $k^2(k+1) \geq (k+1)^2$, for $k \geq 4$, we are done. Divide both sides by $(k+1)$ and we're left with $k^2 \geq (k+1)$, or, equivalently $k^2 - k \geq 1$. This is the same as $k(k-1) \geq 1$. The LHS is at least $4 \cdot 3 = 12$ if $k \geq 4$, so this holds. $\square$

Problem 4. Given a set of any 15 integers, show that one can always choose two of them so that their difference is a multiple of 14.

Proof. Two numbers have a difference divisible by 14 if and only if their residues mod 14 are equal. If we have 15 numbers, there are 15 residues mod 14, but since there are only 14 distinct residues mod 14, we deduce by the pigeon-hole principle, that there must be two equal residues, hence two numbers whose difference is a multiple of 14. ☐

Problem 5. How many permutations of the 26 letters of the English alphabet do not contain any of the strings *fish*, *rat*, or *bird*.

Proof. There are 26! permutations total. Let's count how many permutations contain at least one of the strings *fish*, *rat*, or *bird*. Denote by A, B, C, the sets of permutations which contain *fish*, *rat*, or *bird*, respectively. One may treat a word like *fish* as a block, and then there are 22 other letters, therefore $|A| = 23!$. Similarly, $|B| = 24!$, and $|C| = 23!$. We apply inclusion-exclusion to compute

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Now $A \cap C = \emptyset$, because the letter i would have to appear twice, and also $B \cap C = \emptyset$, because of the letter r. (This implies $A \cap B \cap C = \emptyset$ as well.) Then $|A \cap B| = 21!$, because we treat *fish*, *rat* as two blocks, and there are 19 other letters. In conclusion $|A \cup B \cup C| = 24! + 2 \cdot 23! - 21!$. What we want is the complement, so the answer is

$$26! - 24! - 2 \cdot 23! + 21!$$

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Problem 6. A coin is flipped 10 times where each flip comes up either heads or tails. How many possible outcomes

(a) are there total?
(b) contain exactly two heads?
(c) contain at most three tails?
(d) contain the same number of heads and tails?

Proof.
(a) $2^{10} = 1024$
(b) $C(10,2) = 10 \cdot 9/2 = 45$
(c) $C(10,0) + C(10,1) + C(10,2) + C(10,3) = 1 + 10 + 45 + 105 = 161$
(d) $C(5,5) = \frac{10!}{5!}5! = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 252$ ☐

Problem 7.

(a) What is the coefficient of $x^{101}y^{99}$ in the expansion of $(2x - 3y)^{200}$?

(b) The row of Pascal's triangle containing the binomial coefficients $C(10, k)$, $0 \leq k \leq 10$, is:

$$1 \quad 10 \quad 45 \quad 120 \quad 210 \quad 252 \quad 210 \quad 120 \quad 45 \quad 10 \quad 1$$

Produce the row immediately following this row in Pascal's triangle.

Proof.

(a) $C(200, 101) \cdot 2^{101} \cdot (-3)^{99} = -\frac{200!}{101!99!} \cdot 2^{101} \cdot 3^{99}$

(b)

$$1 \quad 11 \quad 55 \quad 165 \quad 330 \quad 462 \quad 462 \quad 330 \quad 165 \quad 55 \quad 11 \quad 1$$

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