Math 2200-1. Quiz 2. Solutions. Fall 2008.

November 25, 2008

Problem 1. Let f_n, $n \geq 1$ denote the Fibonacci numbers. Recall that they are defined recursively as follows: $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$, for $n \geq 2$.

(1) Find f_7.

(2) Prove by induction that

$$f_1 + f_2 + \cdots + f_n = f_{n+2} - 1,$$

for all $n \geq 1$.

Proof. (1) $f_7 = 13$.

(2) The base case is $n = 1$. The left hand side is $f_1 = 1$ and the right hand side is $f_3 - 1 = 2 - 1 = 1$.

Assume that $f_1 + f_2 + \cdots + f_k = f_{k+2} - 1$. Add f_{k+1} on both sides and get

$$f_1 + f_2 + \cdots + f_k + f_{k+1} = f_{k+2} + f_{k+1} - 1 = f_{k+3} - 1,$$

where the last step uses the recursive formula for $n = k + 3$.

□

Problem 2. Prove by induction that

$$2n + 3 \leq 2^n,$$

for all $n \geq 4$.

Proof. The base case is $n = 4$: $11 \leq 14$, true.

Assume $2k+3 \leq 2^k$, and we want to prove that $2(k+1)+3 \leq 2^{k+1}$, or equivalently $2k+5 \leq 2^{k+1}$. Multiply the inequality in the inductive hypothesis by 2, and we get that $4k + 6 \leq 2^{k+1}$. If we prove that $2k + 5 \leq 4k + 6$, then we are done. But this is equivalent with $2k \geq -1$, which is true for any positive k, in particular for $k \geq 4$.

□

Problem 3. Prove by induction that 3 divides $n^3 + 2n$ for all positive integers n.

Proof. The base case is $n = 1$ when $n^3 + 2n = 3$ so it is divisible by 3.

We assume that 3 divides $k^3 + 2k$, and prove that 3 divides $(k + 1)^3 + 2(k + 1)$. Using the binomial theorem,

$$(k + 1)^3 + 2(k + 1) = k^3 + 3k^2 + 3k + 1 + 2k + 2 = (k^3 + 2k) + 3(k^2 + k + 1).$$

Both terms are divisible by 3, so the claim follows. □

Problem 4. How many permutations of the letters $ABCDEFGH$:

(1) contain the string CDF?
(2) contain the string FH?
(3) contain the string EF?
(4) do not contain any of the strings CDF, FH, and EF?

Proof. (1) We regard CDF as a block, and there are 5 other letters, so 6! permutations.
(2) 7!
(3) 7!
(4) We need to use inclusion-exclusion. Let a, b, c denote the sets in (1), (2), (3), respectively. We compute $|a \cup b \cup c|$:

$$\begin{aligned}|a \cup b \cup c| &= |a| + |b| + |c| - |a \cap b| - |a \cap c| - |b \cap c| + |a \cap b \cap c| \\ &= 6! + 7! + 7! - 5! - 0 - 6! + 0 = 2 \cdot 7! - 5!\end{aligned}$$

We want the complement of $a \cup b \cup c$, so the answer is $8! - 2 \cdot 7! + 5! = 30360$. □

Problem 5.

(1) According to wikipedia, the population of New York State is 19 million. What must be the minimum number of people in the state of New York with the same three initials who were born on the same day of the year (but not necessarily in the same year)? Assume that everyone has three initials. Explain your answer.

(2) Assume all the license plates in NY consist of three letters followed by three digits. How many different license plates are possible if the requirement is that a license plate contains no letter twice *or* no digit twice?

Proof. (1) There are 26^3 possible three initials, and 366 possible birthdays. By the pigeon-hole principle, the answer is $\lceil \frac{19 \cdot 10^6}{26^3 \cdot 366} \rceil = 3$.
(2) Let A be the set of licences with no letter twice and B the set with no digit twice. We want $|A \cup B|$. By inclusion-exclusion:

$$|A \cup B| = |A| + |B| - |A \cap B| = 26 \cdot 25 \cdot 24 \cdot 10^3 + 26^3 \cdot 10 \cdot 9 \cdot 8 - 26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 = 17022720.$$

□

Extra credit problem. *Attempt to solve it only if you finished the first five problems!*
Show that in any set of $n + 1$ positive integers, not exceeding $2n$ there must be two that are relatively prime.

Proof. In the first $2n$ positive integers, there are n pairs of consecutive numbers: $\{1, 2\}, \{2, 3\}, \ldots, \{2n-1, 2n\}$. Since we choose $n + 1$ numbers, by the pigeon-hole principle, two of these numbers must be consecutive. In particular, they are prime to each other. □