Math 2200-1. Quiz 1. Solutions. Fall 2008.

Problem 1. (15 points) Show carefully that the compound proposition

$$((p \vee q) \wedge \overline{p}) \to q$$

is a tautology.

Proof. We use the laws of propositional logic (also a table of truth values would work).

$$((p \vee q) \wedge \overline{p}) \to q \equiv \overline{((p \vee q) \wedge \overline{p})} \vee q \equiv (\overline{(p \vee q)} \vee \overline{\overline{p}}) \wedge q$$
$$\equiv \overline{(p \wedge q)} \wedge (p \wedge q) \equiv \mathsf{T}.$$

We used the resolution of the implication, one of deMorgan laws, the double negation, associativity, and $\overline{r} \wedge r \equiv \mathsf{T}$. □

Problem 2. (15 points) Is the following argument valid? (Carefully express it in propositional logic, and show the rules of inference used at each step.)

"You can score well in the GRE only if you have good analytical skills. Every student who takes Discrete Math has good analytical skills or good memory. Maggie doesn't have good memory. Therefore, if Maggie takes Discrete Math, then Maggie will score well in the GRE."

Proof. Define the following propositional functions, where the domain of the variable x is "all students":

- $P(x) = x$ scores well in the GRE;
- $Q(x) = x$ has good analytical skills;
- $R(x) = x$ takes Discrete Math;
- $S(x) = x$ has good memory.

Then the premises of the argument are:

(a) $\forall x(P(x) \to Q(x))$;
(b) $\forall x(R(x) \to Q(x) \vee S(x))$;
(c) $\overline{S}$(Maggie),

and the conclusion is

(d) R(Maggie) $\to P$(Maggie).

We can use instantiation for (a) and (b) to get

(a') P(Maggie) $\to Q$(Maggie);
(b') R(Maggie) $\to Q$(Maggie) $\vee S$(Maggie).

From (b') and (c), we get

(b") R(Maggie) $\to Q$(Maggie).

But (a) and (b") do not imply the conclusion (d). The argument is invalid. □

Problem 3. (17 points)

(a) Express the following sentence using quantifiers (and logical propositions): *"There is a student in this class who has taken some course in every department in the school of science."*

You should use the variables: s student, c course, d department, and these should be the only variables. Give the domain of each variable.

(b) Find the logical negation of the quantifier expression that you obtained in (a).

(c) Translate the negation you obtained in (b) into an English sentence.

Proof. (a)

Variable	Domain
s student	this class
c course	all courses
d department	departments in the school of science

We define the propositional function:

$P(s, c, d) = s$ took course c in department d.

Then the sentence can be written as

$$\exists s \forall d \exists c \; P(s, c, d).$$

(b) The negation is

$$\forall s \exists d \forall c \overline{P}(s, c, d).$$

(c) In English: "Every student in this class has not taken any course in some department in the school of science."

□

Problem 4. (18 points)

(a) List the elements of the set $S = \{x | \; x \in \mathbb{Z} \text{ and } (x+1)(2-x) \geq 0\}$.

(b) For every integer $i \geq 1$, set $A_i = \{-i, \dots, -1, 0, 1, 2, 3, \dots\}$. Find

$$A_{100} - A_{96}, \qquad \bigcup_{i=1}^{\infty} A_i, \qquad \bigcap_{i=1}^{\infty} A_i.$$

Proof. (a) If $(x+1)(2-x) \geq 0$, then $-1 \leq x \leq 2$. Since x is also an integer, we have $S = \{-1, 0, 1, 2\}$.

(b) $A_{100} - A_{96} = \{-100, -99, -98, -97\}$.

$\bigcup_{i=1}^{\infty} A_i = \mathbb{Z}$, since every nonnegative integer is in every A_i, and if n is a negative integer, then $n \in A_{-n}$.

$\bigcap_{i=1}^{\infty} A_i = A_1 = \{-1, 0, 1, 2, 3, \dots\}$, since A_1 is a subset of every A_i.

□

Problem 5. (18 points)

(a) Is the function $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$, given by $f(x) = 3x^2 + 2$ one-to-one? What is its image(range)?

(b) Consider $g : \mathbb{R} \to \mathbb{Z},\ g(x) = \lfloor x + \frac{1}{2} \rfloor$. What is the preimage $g^{-1}(\{-1, 1\})$?

Proof. (a) f is one-to-one (note that the domain is positive numbers). To see this, prove the contrapositive: $f(x_1) = f(x_2) \iff 3x_1^2 + 2 = 3x_2^2 + 2 \iff x_1^2 = x_2^2$, and since $x_1 > 0, x_2 > 0$, this is equivalent to $x_1 = x_2$.

For the image, set $y = 3x^2 + 2$, then $x^2 = \frac{y-2}{3} > 0$, so $y > 2$. The image is the interval $(2, \infty)$. (In particular, f is not onto.)

(b) Recall that $\lfloor y \rfloor \le y < \lfloor y \rfloor + 1$, for every real number y. Then $\lfloor x + \frac{1}{2} \rfloor = -1$, if and only if $-1 \le x + \frac{1}{2} < 0$, or, equivalently $-\frac{3}{2} \le x < \frac{1}{2}$.

Similarly $\lfloor x + \frac{1}{2} \rfloor = 1$ if and only if $\frac{1}{2} \le x < \frac{3}{2}$.

Therefore, the preimage $g^{-1}(\{-1, 1\}) = [-\frac{3}{2}, -\frac{1}{2}) \cap [\frac{1}{2}, \frac{3}{2})$.

□

Problem 6. (17 points) Prove the following statements:

(a) For every $x > 0$, if x is irrational, then $\sqrt{x}$ is irrational.

(b) Between any two distinct rational numbers, there exist infinitely many rational numbers.

Proof. (a) (proof by contrapositive) If $\sqrt{x}$ is rational, then x is rational. Write $\sqrt{x} = \frac{m}{n}$, for some integers $m, n,\ n \ne 0$. Then by squaring both sides, $x = \frac{m^2}{n^2}$. Since $n \ne 0$, then $n^2 \ne 0$, moreover, both m^2, n^2 are integers, since m, n are. This proves that x is rational.

(b) (proof by contradiction) Assume that there exist two rational numbers $a, b,\ a < b$, such that between a and b there are only finitely many rational numbers. Then there must be a rational number closest to a among them, call it r. We have $a < r$ and no rational numbers between a and r. Consider $\frac{a+r}{2}$. This is a rational number, and $a < \frac{a+r}{2} < r$. This gives the contradiction.

□