Review Problem Set 2 (Vector Spaces)

Name: **ID:**

Code Name:

Instructions

1. Print your name, ID number, and code name above. Your code name will be used for posting grades on the course web page. If you do not supply a code name, your grades will not be posted.
2. If you supplied a code name in the last Test, please leave the code name entry blank.
3. Time allowed: 1 hour.
4. Solve all problems and show all work for full credit.
5. No calculators are allowed.
6. The total number of points in this test is 100. It will be graded out of 100.

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1. Suppose $\mathcal{E} = (\vec{\mathbf{e}}_1, \ldots, \vec{\mathbf{e}}_3)$ and $\mathcal{F} = (\vec{\mathbf{f}}_1, \ldots, \vec{\mathbf{f}}_2)$ are bases for $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively. Let $T \in \mathrm{L}(\mathbb{R}^3, \mathbb{R}^2)$ be given by:
$$T(\alpha_1\vec{\mathbf{e}}_1 + \alpha_2\vec{\mathbf{e}}_2 + \alpha_3\vec{\mathbf{e}}_3) := (11\,\alpha_1 + 12\,\alpha_2 + 13\,\alpha_3)\,\vec{\mathbf{f}}_1 + (21\,\alpha_1 + 22\,\alpha_2 + 23\,\alpha_3)\,\vec{\mathbf{f}}_2.$$
Find $\mathrm{M}_{\mathcal{F}}(T(\mathcal{E}))$.

2. a) Give an example of an infinite-dimensional vector space and prove that it indeed has infinite dimension.
   b) Give an example of a non-linear map between two vector spaces.

3. a) What is the dimension of $\mathbb{R}^{m\times n}$ as a vector space over $\mathbb{R}$?
   b) What is the relation between $\dim \mathrm{L}(\mathbb{R}^n, \mathbb{R}^m)$ and $\dim(\mathbb{R}^{m\times n})$?
   c) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a non-zero linear map. Find $\dim(\ker(f))$.

4. Let $T \in \mathrm{L}(U, V)$ and $S \in \mathrm{L}(V, W)$, where U, V, W are finite-dimensional vector spaces, and let $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ be bases, respectively, for U, V, W. Suppose
$$\mathrm{M}_{\mathcal{F}}(T(\mathcal{E})) = \begin{bmatrix} -3 & 1 & -3 & 1 \\ -2 & 1 & -2 & 0 \\ 2 & 1 & -1 & 0 \end{bmatrix}, \quad \text{and} \quad \mathrm{M}_{\mathcal{G}}(S(\mathcal{F})) = \begin{bmatrix} -2 & -3 & -2 \\ 2 & -2 & -1 \end{bmatrix}.$$
Find $\mathrm{M}_{\mathcal{G}}(S \circ T(\mathcal{E}))$.

5. Prove or disprove the linear independence of the following vectors:
$$\begin{bmatrix} 1 \\ -1 \\ -4 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} -4 \\ -2 \\ -4 \\ -2 \end{bmatrix}, \quad \begin{bmatrix} 6 \\ 0 \\ -4 \\ 8 \end{bmatrix}.$$

6. Prove that the image of a linear map is always a vector subspace of the codomain vector space.

7. Find a basis for the kernel of the linear map $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ defined by:
$$f\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} := \begin{bmatrix} 1 & -2 & 1 & -4 \\ 0 & -5 & -2 & 2 \\ 2 & 1 & 4 & -10 \end{bmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

8. Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) := \begin{bmatrix} 3x + 2y - 4z \\ x - 5y + 3z \end{bmatrix}$$

Let $\mathcal{E}$ and $\mathcal{B}$ be two bases for $\mathbb{R}^3$ given by:

$$\mathcal{E} := \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad \text{and} \quad \mathcal{B} := \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Let $\mathcal{F}$ and $\mathcal{C}$ be two basis for $\mathbb{R}^2$ given by:

$$\mathcal{F} := \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \quad \text{and} \quad \mathcal{C} := \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}.$$

a) Find $\mathrm{M}_{\mathcal{E}}(\mathcal{B})$ and $\mathrm{M}_{\mathcal{F}}(\mathcal{C})$.

b) Find $\mathrm{M}_{\mathcal{F}}(T(\mathcal{E}))$.

c) Find $\mathrm{M}_{\mathcal{C}}(T(\mathcal{B}))$.

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